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Distances for discretely observed jump processes and applications in nonparametric statistics

Joint work with Alexandra Carpentier, Céline Duval, Markus Reiß
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MATHEMATISCHE KOMPLEXITÄTSREDUKTION

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Equivalence in Le Cam sense

Problem: When do two statistical models

$$\mathcal{E}_1 = \big(\mathcal{X}_1, \mathcal{T}_1, (P_{1,\theta} : \theta \in \Theta)\big) \quad \text{and} \quad \mathcal{E}_2 = \big(\mathcal{X}_2, \mathcal{T}_2, (P_{2,\theta} : \theta \in \Theta)\big)$$

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Idea: The statistical models $\mathcal{E}_1$ and $\mathcal{E}_2$ contain "the same amount of information about any $\theta \in \Theta$" if there exist two Markov kernels, K_1 and K_2 that do not depend on θ, such that

$$K_1 P_{1,\theta} = P_{2,\theta} \quad \text{and} \quad K_2 P_{2,\theta} = P_{1,\theta} \quad \forall \theta \in \Theta.$$

Equivalence in Le Cam sense

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Definition

The **Le Cam distance** $\Delta\big(\mathcal{E}_1, \mathcal{E}_2\big)$ is defined as

$$\delta(\mathcal{E}_1, \mathcal{E}_2) = \inf_K \sup_{\theta \in \Theta} \| K P_{1,\theta} - P_{2,\theta} \|_{TV}, \quad \Delta(\mathcal{E}_1, \mathcal{E}_2) = \max(\delta(\mathcal{E}_1, \mathcal{E}_2), \delta(\mathcal{E}_2, \mathcal{E}_1)),$$

where the infimum is taken over all Markov kernels.

Le Cam theory and decision theory

How to transfer estimators: Suppose that $KP_{1,\theta} = P_{2,\theta}$, with $K(x,A) = \mathbf{1}_A S(x)$ for all $x \in \mathcal{X}_1$, $A \in \mathcal{T}_2$ and $\theta \in \Theta$. For any estimator π_2 in $\mathscr{E}_2$ it is possible to define an estimator π_1 in $\mathscr{E}_1$ with the same risk as π_2. Simply take

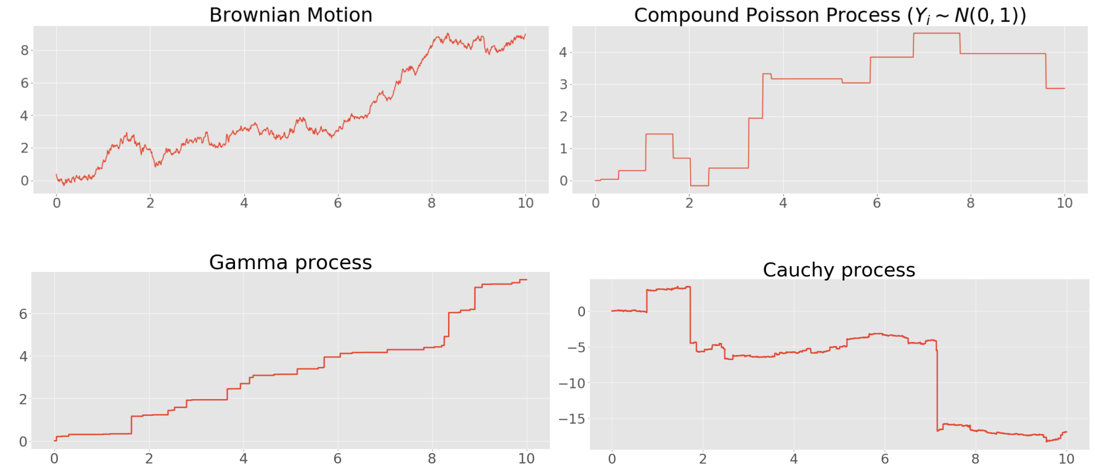
$$\pi_1(x) := \pi_2(S(x)), \forall x \in \mathcal{X}_1.$$

Philosophy: If two models are equivalent then they have the same statistical properties $\Longrightarrow$ it is enough to choose the simplest one when studying these properties.

L. Le Cam, Asymptotic Methods in Statistical Decision Theory, Springer-Verlag, New York. (1986).

Lévy processes

Lévy processes are stochastic processes with independent, stationary increments and trajectories a.s. càdlàg (right continuous with left limits).



Why Lévy processes?

- Lévy processes are the fundamental building blocks in stochastic models with evolution in time exhibiting sudden changes in value.
- They play a central role in many fields of science such as:
  - *mathematical finance*: for modelling market fluctuations;
  - *actuarial science*: for risk theory and premium calculations;
  - *biology*: for modelling the membrane potential;
  - *hydrology*: for modelling rainfall;
  - *physics*: for weather and climate;
  - *seismology*: for earthquakes;
  - *engineering*: to model the functioning of GPS;
  - many others...

Structure of a Lévy process

Lévy process = Gaussian process + small jumps + compound Poisson process.

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Formally, $X \sim (b, \Sigma^2, \nu)$ can be written as:

$$X_t = \Sigma W_t + bt + X_t^S + X_t^B,$$

- W is a standard Brownian motion;
- X^S is a centred martingale describing the small jumps;
- X^B is a compound Poisson process: $X_t^B = \sum_{i=1}^{N_t} Y_i$ with N a Poisson process of intensity $\lambda := \nu(\mathbb{R} \setminus [-1, 1])$ independent of the sequence of i.i.d. $(Y_i)_{i \geq 0}$, $Y_1 \sim \nu_{|\mathbb{R} \setminus [-1,1]} / \lambda$;
- W, X^S and X^B are independent of each other.

Interpretation of the Lévy measure

The jump dynamics of a Lévy process is determined by its Lévy measure ν.
Interpretation of ν: $\forall B \in \mathscr{B}(\mathbb{R} \setminus \{0\})$

$$\nu(B) = \frac{1}{t}\mathbb{E}\Bigg[\sum_{0<s\leq t} \mathbf{1}_B(\Delta X_s)\Bigg] \tag{1}$$

is the average number of jumps whose size is in B.

Example Let $B = (0,1]$ and X be a Lévy process with Lévy density $\frac{\nu(dx)}{dx} = \frac{e^{-x}}{x}\mathbf{1}_{x>0}$. In particular, $\nu(B) = \infty$. Thus, from (1), we deduce that for any time interval $[0,t]$, $(X_s)_{s\in[0,t]}$ will have in expectation infinitely many jumps with size in $(0,1]$.

Lévy density estimation from high frequency data

Let X be a pure jump Lévy process with Lévy measure ν.

Data: $(X_{i\Delta})_{i=1}^n$, with $\Delta = \Delta_n \to 0$ and $n\Delta_n \to \infty$ as $n \to \infty$.

Assumption: ν is absolutely continuous with respect to a dominating measure ν_0. The density $f = \frac{d\nu}{d\nu_0}$ belongs to a nonparametric class $\mathscr{F}$.

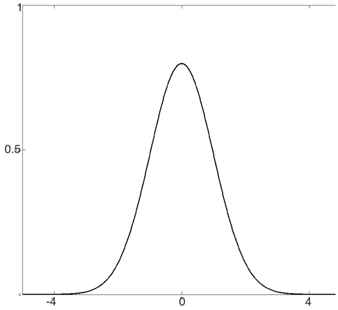
Goal: Build minimax estimators for the Lévy density $f = d\nu/d\nu_0$, i.e. estimators that minimise the minimax risk with respect to a loss function ℓ:

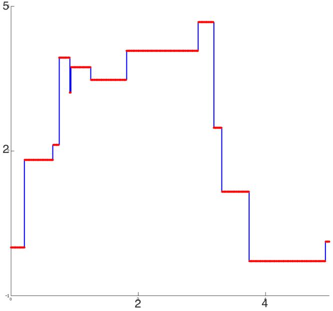
$$R^* := \inf_T \sup_{f \in \mathscr{F}} \mathbb{E}[\ell(T, f)].$$

Remark

1. We face an inverse problem: we do not observe samples of ν but of the law of the increments of X.
2. There is no explicit, clear link between the law of X and ν.
3. ν is an infinite dimensional object. We face a nonparametric estimation problem.
4. If $\nu(\mathbb{R}) = \infty$ and $\nu_0 = Leb$, the Lévy density explodes in a neighborhood of zero.

Illustration: Compound Poisson process $\lambda = 2$, $\Delta = 0.01$

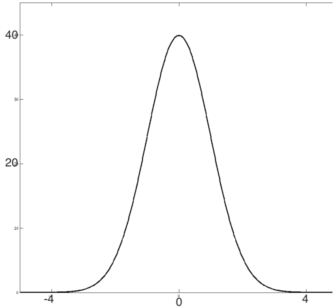




Lévy density: $f(x) := \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$,
$\nu(\mathbb{R}) = 2, \ \nu_0 = Leb$

One trajectory of X

Illustration: Compound Poisson process $\lambda = 100$, $\Delta = 0.01$



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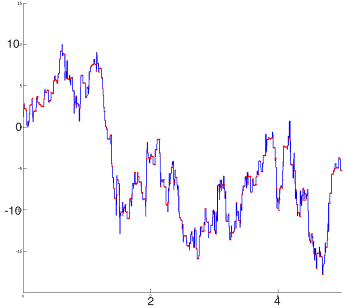

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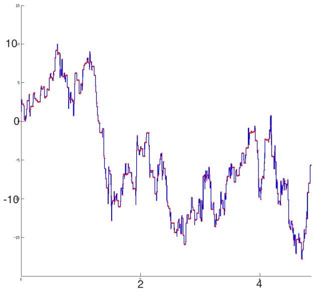

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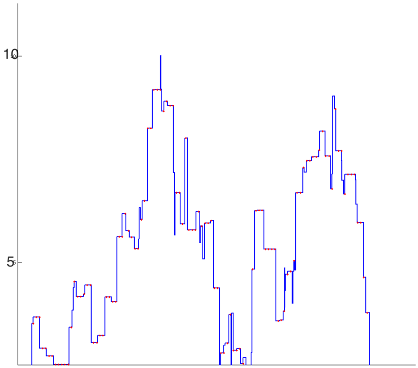

Zoom

Complexity reduction via approximations

$$X = \text{ Gaussian process} + \text{ small jumps } + \text{ compound Poisson process.}$$

- **Le Cam theory:** consists of finding a statistical model, simpler than the Lévy one, and asymptotically equivalent to it for what concerns the estimation of f.
- **Compound Poisson approximation:** consists of ignoring the small jumps. The theoretical justification comes from the fact that any Lévy process is the (weak) limit of a sequence of compound Poisson processes.
- **Gaussian approximation for the small jumps:** consists of proving that the law of the discrete observations of the small jumps strongly converges to the law of a Gaussian vector.

Literature

Case $\nu(\mathbb{R}) < \infty$: well understood.

Case $\nu(\mathbb{R}) = \infty$: more challenging because of the small jumps. The analysis so far relies on spectral approaches (L_2 and L_∞ loss) and one typically estimates functionals of f such as $xf(x)$ and $x^2 f(x)$.

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References (from high- and low-frequency observations):
Compound Poisson process: Buchmann and Grübel, (2003); van Es, Gugushvili, Spreij, (2007); Duval, (2013); Comte, Duval, Genon-Catalot, (2014); Coca, (2018); Gugushvili, M., van der Meulen (2019).

Infinite activity: Comte, Genon-Catalot, (2009), (2010), (2011); Figueroa-López, Houdré, (2009); Neumann, Reiß, (2009); Kappus, Reiß, (2010); Gugushvili, (2012); Trabs, (2015).

Textbook: Lévy matters IV - Estimation for discretely observed Lévy processes. Belomenstny, Comte, Genon-Catalot, Masuda, Reiß, (2015).

Asymptotic equivalence result

$$\mathscr{P}_n: \quad (X_{i\Delta} - X_{(i-1)\Delta})_{i=1}^n, \quad X = X^S + X^B \text{ with } f = d\nu/d\nu_0 \text{ on } I, \quad f \in \mathscr{F},$$

$$\mathscr{W}_n: \quad dy_t = \sqrt{f(t)}dt + \frac{dW_t}{2\sqrt{n\Delta}\sqrt{g(t)}}, \quad g = \frac{d\nu_0}{dx}, \quad t \in I, \quad f \in \mathscr{F}.$$

Theorem (M., 2016)

Suppose that $\Delta = \Delta_n \to 0$, $n\Delta_n \to \infty$ *and* $n\Delta_n^2 \to 0$ *as* $n \to \infty$. *Under appropriate assumptions on* $\mathscr{F}$ *it holds:*

$$\Delta(\mathscr{P}_n, \mathscr{W}_n) \to 0 \quad \text{as} \quad n \to \infty.$$

The upper bound on $\Delta(\mathscr{P}_n, \mathscr{W}_n)$ *is explicit as well as all the equivalence mappings.*

E. Mariucci, Asymptotic equivalence for pure jump Lévy processes with unknown Lévy density and Gaussian white noise. Stoch. Proc. Appl. 126.2 (2016)

Interpretation

It is well known that a density estimation problem with i.i.d. random variables $(X_i)_{i=1}^n$ with support in I and common density $f \in \mathscr{F}$ (with respect to the Lebesgue measure) is asymptotically equivalent to a Gaussian white noise model:

$$dy_t = \sqrt{f(t)}dt + \frac{dW_t}{2\sqrt{n}}, \quad t \in I, \quad f \in \mathscr{F}.$$

$\Longrightarrow$ Estimating the Lévy density from the observation of n equidistant increments of a Lévy process with a sample rate Δ is "as difficult" as estimating the density of $n\Delta$ i.i.d. random variables.

$\Longrightarrow$ If $\mathscr{F}$ is a class of regularity s, then we expect that the minimax rate of convergence for the estimation of the Lévy density $f \in \mathscr{F}$ is $(n\Delta)^{-\frac{sp}{2s+1}}$ for an L_p loss.

Compound Poisson approximation

- Let $I(\varepsilon) := (-\overline{A}, -\varepsilon] \cup [\varepsilon, \overline{A})$, $\varepsilon > 0$ and $\overline{A} \in (\varepsilon, \infty]$. For any $x \in I(\varepsilon)$,

$$f(x) = f(x)\mathbf{1}_{|x|>\varepsilon} = \lambda_\varepsilon h_\varepsilon(x),$$

  where $h_\varepsilon(x) = \frac{f(x)}{\lambda_\varepsilon}\mathbf{1}_{|x|>\varepsilon}$.
- We observe that $\lambda_\varepsilon h_\varepsilon(x)$ is the Lévy density of a CPP with intensity $\lambda_\varepsilon := \nu(x : |x| > \varepsilon)$ and jump density h_ε.
- We define

$$\hat{f}_n(x) := \hat{\lambda}_{n,\varepsilon}\hat{h}_{n,\varepsilon}(x), \quad \forall x \in I(\varepsilon).$$

- We assume $f \in \mathscr{F}(s, p, q, M_\varepsilon)$,

$$\mathscr{F}(s, p, q, M_\varepsilon,) = \left\{f \in L_p(I(\varepsilon)) \ : \ \|f\|_{B^s_{p,q}(I(\varepsilon))} \leq \mathcal{M}_\varepsilon\right\},$$

  where $\mathcal{M}_\varepsilon := \lambda_\varepsilon \mathcal{M}$, $\mathcal{M}$ being fixed, $p \in [2, \infty)$, $q \in [1, \infty]$ and $s > 1/p$.

Main result (in a simplified setting)

Set $\mathscr{F} = \mathscr{F}(s,p,q,M_\varepsilon) \cap \left\{ f : f(x) \leq \frac{M}{|x|^{1+\alpha}} \ \forall |x| \leq 2 \right\} \cap \left\{ f : \int_{|x|\geq\varepsilon} f(x)dx \geq 1 \right\}$.

Theorem (Duval, M.)

Let X be a pure jump Lévy process with Lévy density f. Fix $\varepsilon \in (0,1]$ and suppose that $f \in \mathscr{F}$ for some $p \in [2,\infty)$, $q \in [1,\infty]$, $s \geq 3/2 - 1/p$, $M_\varepsilon, M > 0$ and $\alpha \in (0,1)$. Then, for all $n \geq 1$ and $\Delta > 0$ such that $n\Delta \geq 1$ and $\Delta \leq \mathcal{C}_\alpha \varepsilon^\alpha M^{-1}$, it holds that

$$\sup_{f\in\mathscr{F}} \mathbb{E}\left[\int_{I(\varepsilon)} |\hat{f}_{n,\varepsilon}(x) - f(x)|^p dx \right] \leq C(n\Delta)^{-\frac{sp}{2s+1}},$$

where $C > 0$ is a constant independent of n and Δ.

Remarks

It is possible to have a more general upper bound for $\mathbb{E}\big[\int_{I(\varepsilon)} |\hat{f}_{n,\varepsilon}(x) - f(x)|^p dx\big]$ that holds true for all $\varepsilon \in (0,1]$, $f_\varepsilon \in \mathscr{F}$, $\Delta > 0$, $n\Delta \geq 1$. From such a bound one derives that:

- The estimator $\hat{f}_{n,\varepsilon}$ is consistent as soon as $n\Delta \to \infty$.
- Up to a log-factor, the rate of convergence for $\hat{f}_{n,\varepsilon}$ is still $(n\Delta)^{-\frac{sp}{2s+1}}$ even for infinite variation Lévy processes with a Lévy measure controlled by Mx^{-2} in a neighbourhood of zero and under the assumption $n\Delta^2 \leq 1$.
- However, if the Lévy measure behaves as $M|x|^{-(1+\alpha)}$ with $\alpha \in (1,2)$, then the rate of convergence deteriorates. The rate can be explicitly computed and it depends on α.

Gaussian approximation of the small jumps

Set $\sigma^2(\varepsilon) := \int_{|x|\leq\varepsilon} x^2\nu(dx)$. Then, by a result of Gnedenko and Kolmogorov,

$$\frac{X_t^S(\varepsilon)}{\sqrt{t\sigma^2(\varepsilon)}} \xrightarrow[\varepsilon\to 0]{\mathscr{L}} \mathcal{N}(0,1).$$

B.V. Gnedenko and A.N. Kolmogorov, Limit distributions for sums of independent random variables. Addison-Wesley, 1954.

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- Convergence in law is a weak notion of convergence. Is it possible to show the convergence for a meaningful metric?
- If so, what is the exact order for the rate of convergence?

B.V. Gnedenko and A.N. Kolmogorov, Limit distributions for sums of independent random variables. Addison-Wesley, 1954.

Gaussian approximation for the small jumps in Wasserstein distance

Theorem (M., Reiß)

Let $\varepsilon \in (0,1]$ *and* X *be any Lévy process with Lévy measure* ν. *For any* $p \in [1,2]$, *there exists a positive constant* C, *such that*

$$\mathscr{W}_p\Big(\mathscr{L}(X_t^S(\varepsilon)), \mathcal{N}\big(0, t\sigma^2(\varepsilon)\big)\Big) \leq C \min\left(\sqrt{t}\sigma(\varepsilon), \left(\frac{\int_{-\varepsilon}^{\varepsilon} |x|^{p+2}\nu(dx)}{\sigma^2(\varepsilon)}\right)^{1/p}\right)$$
$$\leq C \min\Big(\sqrt{t}\sigma(\varepsilon), \varepsilon\Big).$$

In particular, for $p = 1$ *the bound is* $\min(2\sqrt{t}\sigma(\varepsilon), \frac{\varepsilon}{2})$.

E. Mariucci, M. Reiß, Wasserstein and total variation distance between marginals of Lévy processes, Electronic Journal of Statistics, 2018.

Gaussian approximation for the small jumps in total variation

Theorem (Carpentier, Duval, M.)

Let $\varepsilon \in (0,1]$ *and* X *be a pure jump Lévy process with Lévy measure* ν. *Set* $\nu_\varepsilon := \nu_{|_{[-\varepsilon,\varepsilon]}}$, $\mu_3(\varepsilon) = \int_{-\varepsilon}^{\varepsilon} x^3 \nu_\varepsilon(dx)$ *and* $\mu_4(\varepsilon) = \int_{-\varepsilon}^{\varepsilon} x^4 \nu_\varepsilon(dx)$. *Suppose that*

$$\frac{c_-}{|x|^{1+\alpha}} \le \frac{\nu_\varepsilon(dx)}{dx} \le \frac{c_+}{|x|^{1+\alpha}}, \quad \text{for some } c_+, c_- > 0, \ \alpha \in (0,2).$$

Then, there exists a constant $C > 0$ *such that for all* $n \ge 1$ *and* $\Delta > 0$ *with* $n\Delta \ge 1$ *it holds:*

$$\left\| \mathscr{L}(X_\Delta^S)^{\otimes n} - \mathcal{N}(0,\Delta(\sigma^2(\varepsilon))^{\otimes n} \right\|_{TV} \le C\left(\sqrt{\frac{n\mu_4^2(\varepsilon)}{\Delta^2(\sigma^2(\varepsilon))^4} + \frac{n\mu_3^2(\varepsilon)}{\Delta(\sigma^2(\varepsilon))^3}} + \frac{1}{n} \right).$$

A. Carpentier, C. Duval, E. Mariucci, Total variation distance for discretely observed Lévy processes: a Gaussian approximation of the small jumps, arXiv:1810.02998.

Conclusions:

- We consider the problem of the nonparametric estimation of the Lévy density from the high-frequency observations of one trajectory of a Lévy process.
- We prove the asymptotic equivalence between the Lévy density estimation problem and a Gaussian white noise model.
- We propose an estimator of the Lévy density based on a compound Poisson approximation. It achieves (nearly)-optimal minimax rates for Lévy densities in Besov balls that are bounded by $|x|^{-2}$ in a neighbourhood of zero. If the Lévy density behaves as $|x|^{-(1+\alpha)}$ for some $\alpha \in (1,2)$ in a neighbourhood of zero, then then the rates are slower.
- We provide sharp bounds for a Gaussian approximation of the small jumps in Wasserstein and total variation distance.
- The problem of including a Gaussian approximation for the small jumps in the estimation procedure and studying its statistical properties is still open.

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Thank you for your attention!