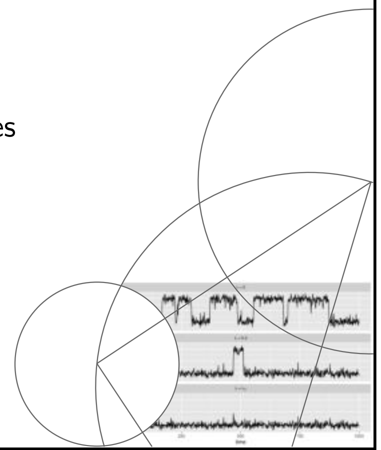
Faculty of Science

Time scales in early warnings: a probabilistic approach

Susanne Ditlevsen
Department of Mathematical Sciences
University of Copenhagen

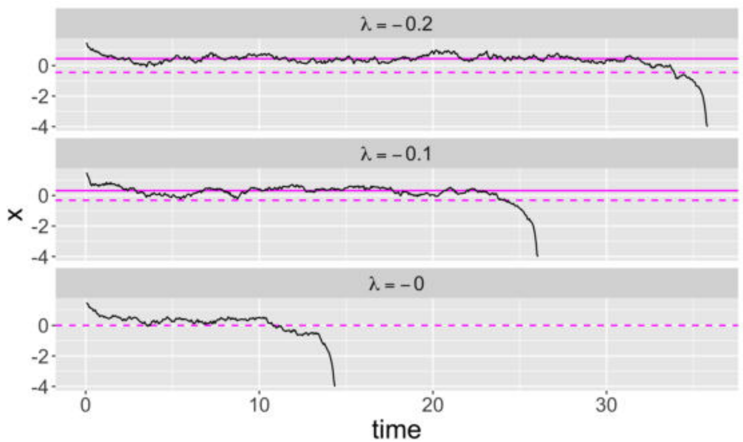
Joint work with Peter Ditlevsen



Tipping points

- Risks of collapse or tipping points exist in a wide variety of complex systems. Climate, medical conditions, pandemics, ecosystems, finance, society, you name it...
- Multiple spatial and temporal scales. Incomplete understanding or uncertainty of the dynamics.
- Predictability is limited. Chaotic nature of the system. Limited resolution in observations.
- Assume dynamics beyond the horizon of prediction as being stochastic. Requires well mixing in state space.
- The stochastic models permit a statistical description.

Tipping points



The goal

- Prediction of the probability of tipping/collapse
- Prediction of the time of tipping
- We need Early Warning Signals!

Early Warning Signals

Control parameter λ

A structural change in the dynamics happens by changing a control parameter λ through a critical value λ_c.

The statistically stable state ceases to exist. The system moves to a different statistically stable state.

The system undergoes a *bifurcation*.

IMPORTANT: For λ sufficiently close to λ_c it can happen in a limited number of ways independent from the details in the governing dynamics.

THEREFORE: Mathematics/stochastics can provide *general* answers valid for *many* systems.

The bifurcation

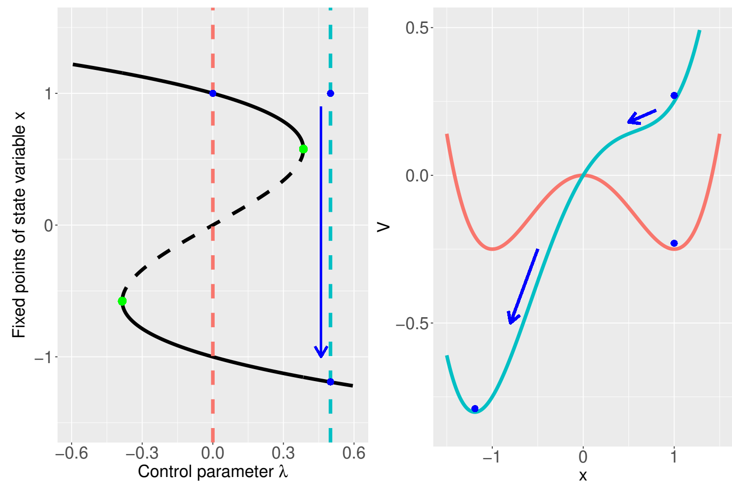
For a codimension-one bifurcation (depending on one single parameter λ) there are two generic bifurcations:

- **Saddle-node bifurcation.** The largest eigenvalue of the Jacobian of the dynamics changes sign.
- **Hopf-bifurcation.** A set of complex conjugate eigenvalues crosses the imaginary axis and the stable state is replaced by a limit cycle oscillation.

Under the saddle-node bifurcation scenario we ask the question:

Under which conditions is it possible to obtain an early warning of a forthcoming critical transition?

The bifurcation diagram determines the dynamics



A simple generic model

$$dX_t = -(X_t^2 + \lambda)dt + \sigma dB_t$$

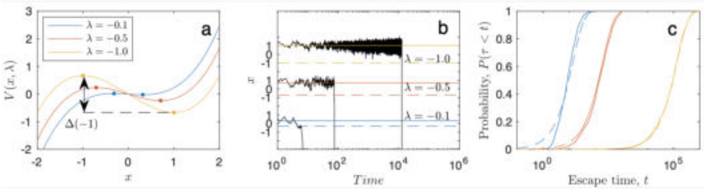
X_t : Representative observable of a system at time t

λ : Control parameter, $\lambda \leq 0$. Might change over time

σ : Amplitude of the noise

B_t : Standard Brownian motion

Risk of a critical transition through a saddle-node bifurcation for $\lambda = 0$.



The control parameter λ

Assume linear ramping

$$\lambda(t) = \lambda_0(1 - \mathbf{1}_{\{t>t_0\}}(t - t_0)/\tau_r)$$

λ_0 : Static value (negative) of control parameter λ

t_0 : Unknown time when λ starts increasing

τ_r : Ramping time constant for the change of λ.

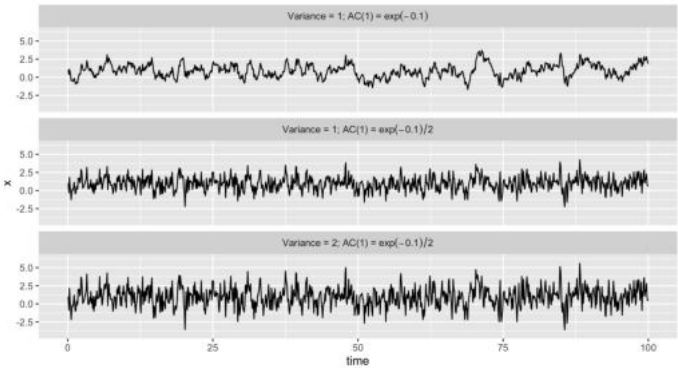
Time of bifurcation equals $t_c := t_0 + \tau_r$ and $\lambda(t_c) = 0$. Tipping happens close to t_c (typically before because of stochastic fluctuations).

QUESTION: How to predict when the tipping happens? Prediction must be based solely on observing the state!

Obviously: If λ_0, t_0 and τ_r are known, the critical transition is perfectly known and predictable.

Early warning signals are statistical

- Increased variance in the fluctuations around the steady state
- Increased autocorrelation of the signal (critical slowing down)



Local approximation with Ornstein-Uhlenbeck process

A linear expansion of the drift term around the steady state $x_+(\lambda)$ yields the approximation

$$X_t = -\alpha(\lambda)(X_t - x_+(\lambda))dt + \sigma dB_t$$

where $\alpha(\lambda) = 2\sqrt{|\lambda|}$ is the mean reversion rate and $x_+(\lambda) = \sqrt{|\lambda|}$ is the stable state.

It has variance

$$\gamma^2 = \sigma^2/2\alpha(\lambda) = \sigma^2/4\sqrt{|\lambda|}$$

and autocorrelation

$$\rho(t) = \exp(-\alpha(\lambda)|t|) = \exp(-2\sqrt{|\lambda|}|t|).$$

Fixed λ: can be estimated by maximum likelihood.

Detection from data

Real time observations of X_t.

A certain observational time window T_{win} is required to detect a significant change in the (quasi-stationary) statistics.

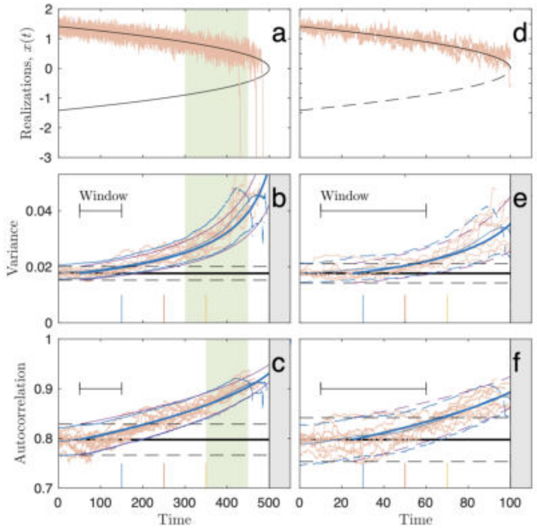
Maximum likelihood theory provides precise estimates. For variance,

$$T_{win} > 2q^2 \left(\frac{\alpha(t)/\sqrt{\alpha_0} + \alpha_0/\sqrt{\alpha(t)}}{\alpha_0 - \alpha(t)} \right)^2,$$

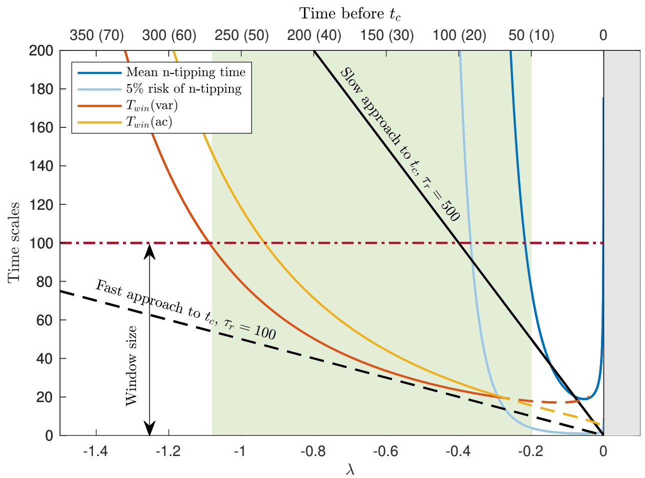
and for autocorrelation,

$$T_{win} > 2q^2 \left(\frac{\sqrt{\alpha_0} + \sqrt{\alpha(t)}}{\alpha_0 - \alpha(t)} \right)^2 \rho_0^{-2}.$$

Detections

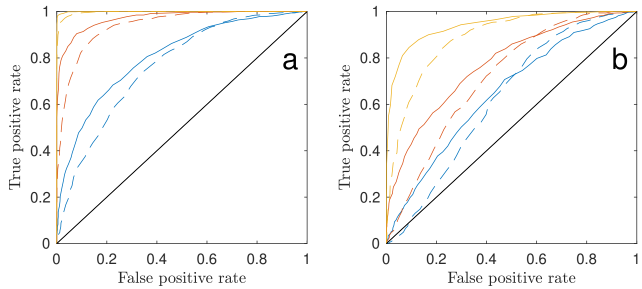


The time scales involved



Can we predict?

ROC curves: variance (solid) or autocorrelation (dashed)



Blue/red/orange curves are for times 0.3/0.5/0.7 τ_r.

Panel a: slow ramping, $\tau_r = 500$.
Panel b: fast ramping, $\tau_r = 100$.

Estimation of the tipping point

The variance at time t is:

$$\gamma_t^2 = \frac{\sigma^2}{2\alpha_t} = \frac{\sigma^2}{4\sqrt{\lambda_t}} = \frac{\sigma^2}{4\sqrt{|\lambda_0|(1 - t/\tau_r)}}$$

We have estimates $\hat{\gamma}_t^2$ at times t, τ_r is unknown.

Define (linear in t!)

$$Y_t = \frac{1}{(\gamma_t^2)^2}; \quad \hat{Y}_t = \frac{1}{(\hat{\gamma}_t^2)^2}.$$

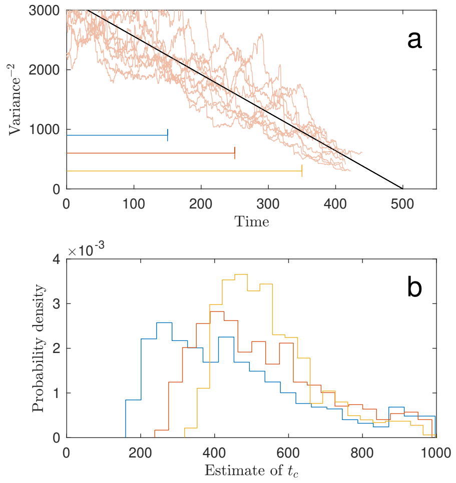
$$Y_t = \frac{4\alpha_t^2}{\sigma^4} = \frac{16|\lambda_0|}{\sigma^4} - \frac{16|\lambda_0|}{\sigma^4 \tau_r} t = \beta_0 + \beta_1 t$$

Then

$$\tau_r = -\frac{\beta_0}{\beta_1}; \quad \hat{\tau}_r = -\frac{\hat{\beta}_0}{\hat{\beta}_1}$$

is (an estimator of) the time of tipping.

Estimation of the tipping point



Estimation of the tipping point

$$\hat{\tau}_r = \frac{\hat{\beta}_0}{\hat{\beta}_1}$$

where $\hat{\beta}_0, \hat{\beta}_1$ are estimates obtained by a weighted linear regression of $\hat{Y}_t$ on time.

The variance of $\hat{Y}_t$ depends on $\alpha(t)$ (therefore weighted).

PROBLEM: $\hat{Y}_{t_i}$ and $\hat{Y}_{t_j}$ are not independent, the usual standard errors are not valid.

Use only time points $t_i = T_{win}(i - 1/2), i = 1, \ldots, k$ to compute more realistic standard errors.

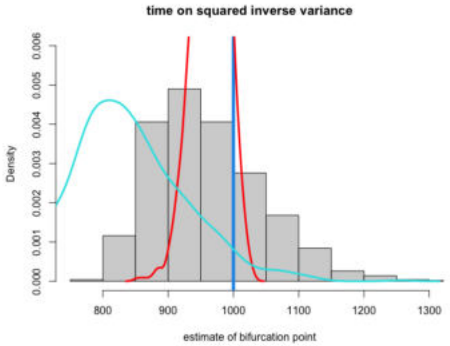
Estimation of the tipping point

$$\hat{\tau}_r = \frac{\hat{\beta}_0}{\hat{\beta}_1}$$

ANOTHER PROBLEM: Statistically, the ratio is very unstable...

SOLUTION: Make the inverse regression: time on $1/(\hat{\gamma}_t^2)^2$. Then the intercept is an estimate of τ_r. Good statistical properties!

Estimation of the tipping point



True tipping (bifurcation) point: $\tau_r = 1000$

Average 95% confidence limits: $(796, 1128)$. Coverage: 76%

Average 5% risk of tipping: 823. Coverage: 96%

Sea Surface Temperature anomaly, subpolar gyre area in the North Atlantic