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Uncertainty Quantification

Multilevel Monte Carlo Methods for Parametric Expectations: Distribution and Robustness Measures

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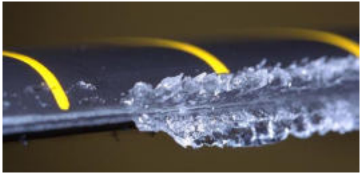
joint work with: Q. Ayoul-Guilmard, S. Ganesh, and F. Nobile

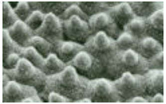
Uncertainty Quantification

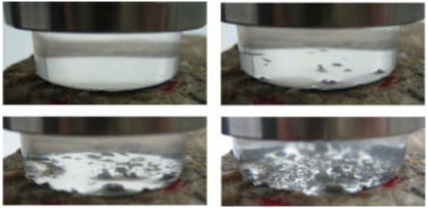
Mathematical models describing real-word phenomena *are affected by* uncertainties and incomplete knowledge.

- **Aerodynamic design**



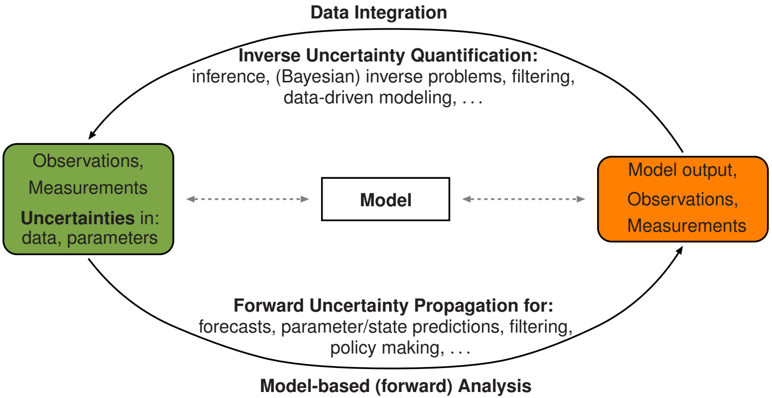
- **Rough contact mechanics** (Lotus effect, rubber on concrete, . . .)





From models to data and back

Reliably using the models requires the study of the impact of all forms of error and uncertainty in models.



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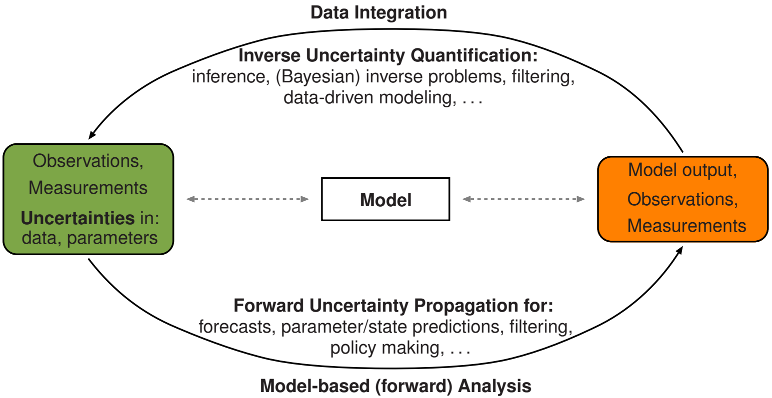
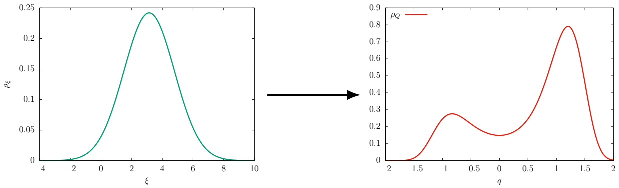


Today's focus: Forward Uncertainty Quantification — quantifying the effects of uncertainties on model predictions induced by model/input uncertainties

Forward Propagation of Uncertainties

random input: ξ $\longrightarrow$ Model $\longrightarrow$ quantity of interest: $Q \equiv Q(\xi)$

- Random input parameters ξ, possibly infinite dimensional
- **Model**: for ξ given, find u s.t. $\mathcal{M}_{\xi}(u) = 0 \rightsquigarrow$ **Quantity of interest:** $Q(\xi) = Q\big(u(\xi)\big)$



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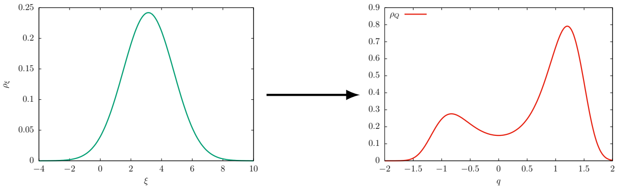

- **Goal:** compute statistical quantities of Q with minimal regularity assumptions.

Forward Propagation of Uncertainties

random input: $\boldsymbol{\xi}$ → **Model** → quantity of interest: $Q \equiv Q(\boldsymbol{\xi})$

$\downarrow$ approximations

computational Model → $Q_h \equiv Q_h(\boldsymbol{\xi})$

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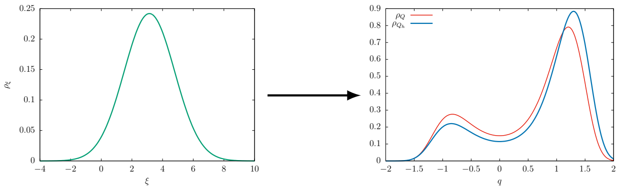

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Monte Carlo method

- General purpose method to approximate moments of Q_h for *given discretization h*:
  - generate sample of N i.i.d. realizations $\xi^{(1)}, \dots, \xi^{(N)} \sim \mathbb{P}_\xi$
  - compute corresponding model outputs $Q_h\big(\xi^{(i)}\big)$, $i = 1, \dots, N$
  - approximate expectation of Q_h by **sample average**:

$$\mathbb{E}(Q) \approx \mathbb{E}(Q_h) \approx E_N(Q_h) := \frac{1}{N}\sum_{i=1}^{N} Q_h^{(i)} \,, \quad Q_h^{(i)} \equiv Q_h\big(\xi^{(i)}\big) \text{ i.i.d. realizations}$$

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- That is, the parameters are implied by MSE tolerance with:
  - h dictated by the bias = discretization error of *approximate model*
  - sample size N given by the statistical error

Monte Carlo: cost vs. accuracy

Lemma: MSE complexity analysis MC (error vs. cost)

Suppose that:

- $\left|\mathbb{E}(Q - Q_h)\right| = \mathcal{O}(h^\alpha)$,
- $\mathrm{Var}(Q_h) = \mathcal{O}(1)$,
- $\mathrm{cost}\left({Q_h}^{(i)}\right) = \mathcal{O}(h^{-\gamma})$.

Achieving $\mathrm{MSE} \leq \varepsilon^2$ requires $N = \mathcal{O}(\varepsilon^{-2})$ and $h = \mathcal{O}(\varepsilon^{1/\alpha})$, resulting in

$$\text{comp. cost MC} = N \,\mathrm{cost}\left(Q_h\right) = \mathcal{O}\left(\varepsilon^{-(2+\gamma/\alpha)}\right) \,.$$

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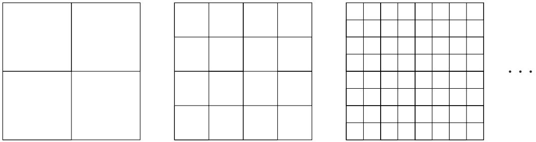
- Another error criterion is the **probability of failure**: find (h, N) such that

$$\mathbb{P}\big(|\mathbb{E}(Q) - E_N(Q_h)| \geq 2\varepsilon\big) \leq \tau , \quad \text{for } \varepsilon > 0 \text{ and } \tau \in (0, 1) .$$

- For $\varepsilon \ll 1$, the distribution of $E_N(Q_h)$ can be approximated in view of central limit theorem (CLT) as $\mathrm{Var}(Q_h)$ is bounded.

Multilevel Monte Carlo for expected values

- **Main idea:** use a hierarchy of $L \in \mathbb{N}_0$ discretizations $h_0 > h_1 > \cdots > h_L$, instead of just one fine level:



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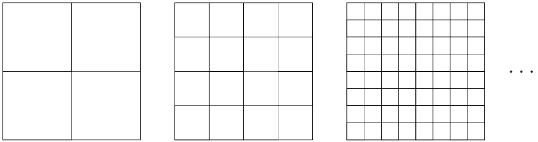

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- The multilevel Monte Carlo (MLMC) estimator of $\mathbb{E}(Q)$ then is ($Q_\ell \equiv Q_{h_\ell}$):

$$\mathbb{E}(Q) \approx \mathbb{E}(Q_L) = \mathbb{E}(Q_0) + \sum_{\ell=1}^{L} \mathbb{E}(Q_\ell - Q_{\ell-1}) \approx E_{N_0}(Q_0) + \sum_{\ell=1}^{L} E_{N_\ell}(Q_\ell - Q_{\ell-1}) =: \hat{Q}_{N,L}$$

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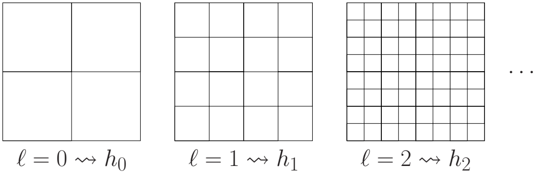

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- MSE accuracy of multilevel Monte Carlo method is

$$\mathrm{MSE} \equiv \mathbb{E}\left(\left|\mathbb{E}(Q) - \hat{Q}_{N,L}\right|^2\right) = \underbrace{\left(\mathbb{E}(Q) - \mathbb{E}(Q_L)\right)^2}_{\text{squared bias}} + \underbrace{\sum_{\ell=0}^{L} \frac{\mathrm{Var}(Q_\ell - Q_{\ell-1})}{N_\ell}}_{\text{statistical error}}, \quad Q_{-1} \equiv 0$$

- Select parameters $L \in \mathbb{N}_0$ and $N \equiv (N_0, N_1, \ldots N_L)^T \in \mathbb{N}^{L+1}$ s.t. MSE criterion is met at minimal cost.
- Optimal parameters by **balancing errors:**
  - bias = discretization error $\rightsquigarrow L$,
  - **determine** N_ℓ by minimizing $\sum_{\ell=1}^{L} N_\ell C_\ell$ subject to MSE $\leq \varepsilon^2$,
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Theorem: Complexity analysis MLMC for $h_{\ell-1}/h_\ell = s > 1$ [Giles, 2008; Cliffe et al., 2011]

Suppose that:
- $\left|\mathbb{E}(Q - Q_\ell)\right| = \mathcal{O}({h_\ell}^{\alpha})$,
- $\mathrm{Var}(Q_\ell - Q_{\ell-1}) = \mathcal{O}({h_\ell}^{\beta})$,
- $C_\ell = \mathcal{O}({h_\ell}^{-\gamma})$.

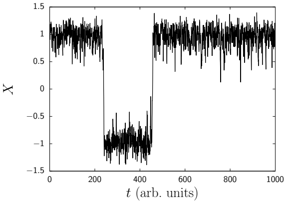
If $2\alpha \geq \min\{\beta, \gamma\}$, then there exists an MLMC estimator $\hat{Q}_{N,L}$ that satisfies MSE $\leq \varepsilon^2$ with

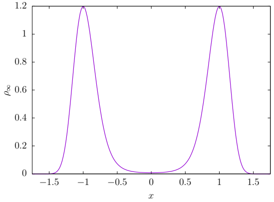
$$\text{comp. cost MLMC} = \sum_{\ell=1}^{L} N_\ell C_\ell \lesssim \begin{cases} \varepsilon^{-2}, & \beta > \gamma, \\ \varepsilon^{-2} \ln(\varepsilon)^2, & \beta = \gamma, \\ \varepsilon^{-\left(2 + \frac{\gamma-\beta}{\alpha}\right)}, & \beta < \gamma. \end{cases}$$

- Other error criteria are possible, e.g., tuning MLMC for *probability of failure* is possible thanks to CLT [Collier et al. 2015], [Hoel, K. 2019].

Assessing the distribution beyond moments

- Expectation is a **good starting point**, but we would like more: **assessing the distribution**





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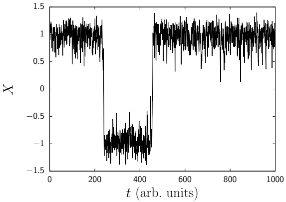

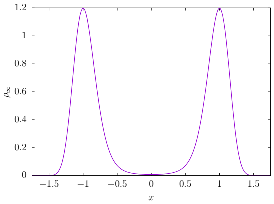

- One could estimate central moments of arbitrary order

$$\mu_p(Q) := \mathbb{E}\left[\left(Q - \mathbb{E}(Q)\right)^p\right], \quad p \in \mathbb{N}$$

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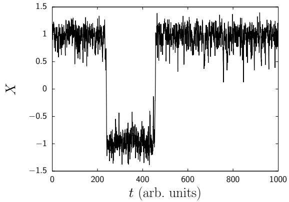

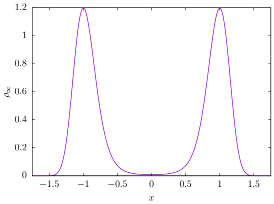

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- However, some applications require techniques that systematically go **beyond using a few moments**: characterization of entire distribution or applications in risk averse optimization, e.g., involving quantiles

$$\arg\min_{z\in\mathbb{R}^d} \mathcal{R}(z) , \quad \mathcal{R}(z) \equiv \mathcal{R}(z;\mathbb{P}_Q) : \text{"risk" of design parameter } z$$

Parametric Expectations

- Many such distributional assessments are related to **parametric expectations** of the form

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Examples:

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1. **Characteristic function** of Q: $\phi(\vartheta, Q) = e^{i\vartheta Q} = \cos(\vartheta Q) + i \sin(\vartheta Q)$
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Difficulty for CDF: $\phi(\vartheta, \cdot)$ is **discontinuous**! Consequently, the variance $\operatorname{Var}\big(\phi(\theta_j, Q_\ell) - \phi(\theta_j, Q_{\ell-1})\big)$ will decay slowly $\rightsquigarrow$ Not much gain in MLMC.

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Remedies in the context of MLMC (cf. [Giles. 2023]):

- Regularizing ϕ to yield $\Phi(\vartheta) = \mathbb{E}\big(\phi(\vartheta, Q)\big) \approx \mathbb{E}\big(\phi_\delta(\vartheta, Q)\big)$ [Giles, Nagapetyan, Ritter. 2015]
- Approximate CDF (PDF) based on MLMC estimates of moments [Bierig, Chernov. 2016]
- numerical smoothing via conditioning [Bayer, Ben Hammouda, Tempone. 2022]

Remedy for CDF: antiderivative/integration approach [K., Nobile. 2018]

For any $\tau \in (0,1)$ define

$$\Phi(\vartheta) = \mathbb{E}\big(\phi(\vartheta, Q)\big) \;, \quad \phi(\vartheta, Q) = \vartheta + \frac{1}{1-\tau}(Q-\vartheta)^+ \;.$$

Then

$$F(\vartheta) = (1-\tau)\Phi'(\vartheta) + \tau \;,$$

which is the **starting point for the MLMC estimator**:

$\rightsquigarrow$ requires effective estimator of Φ and its derivatives.

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This approach may also offer approximations for:

- PDF: $f(\vartheta) = F'(\vartheta) = (1-\tau)\Phi''(\vartheta)$
- τ-quantile: $q_\tau = \inf\{\vartheta\colon F(\vartheta) \geq \tau\} = \arg\min_{\vartheta\in\mathbb{R}} \Phi(\vartheta)$
- Conditional Value at Risk: $\mathrm{CVaR}_\tau = \frac{1}{1-\tau}\int_{q_\tau}^{\infty} \vartheta\, dF(\vartheta) = \min_{\vartheta\in\mathbb{R}} \Phi(\vartheta)$

MLMC method for parametric expectations

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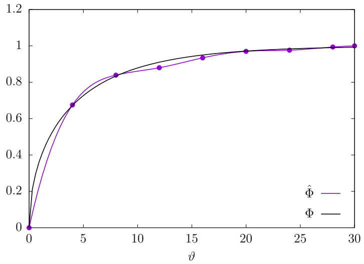

Interpolation approach (e.g., splines or polynomials):

- introduce grid $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n) \subset \Theta$
- compute MLMC estimate $\Phi_L^{\mathrm{MLMC}}(\theta_j)$ of $\Phi(\theta_j)$, $j = 1, \ldots, n$ (same samples of Q_ℓ for every θ_j)
- interpolate values $\Phi_L^{\mathrm{MLMC}}(\boldsymbol{\theta}) = \big(\Phi_L^{\mathrm{MLMC}}(\theta_j)\big)_{j=1}^n$:
  $\hat{\Phi}_L = \mathcal{I}_n\big(\Phi_L^{\mathrm{MLMC}}(\boldsymbol{\theta})\big)$

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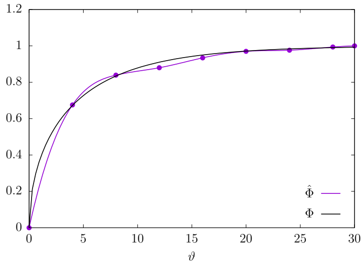

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Exemplary **a-priori properties** for Spline interpolation $\mathcal{I}_n$:

- $\Big\| f^{(m)} - \frac{d^m}{d\vartheta^m}\mathcal{I}_n\big(f(\boldsymbol{\theta})\big)\Big\|_{L^\infty} \leq c_1 n^{m-(k+1)}$, for any $f \in C^{k+1}(\bar{\Theta})$, $m \leq k$
- $\big\|\mathcal{I}_n(\boldsymbol{x})\big\|_{L^\infty} \leq c_2 \|\boldsymbol{x}\|_{\ell^\infty}$, for any $\boldsymbol{x} \in \mathbb{R}^n$
- $\text{cost}\big(\mathcal{I}_n(\boldsymbol{x})\big) \leq c_3 n$, $\boldsymbol{x} \in \mathbb{R}^n$
- $\Big\|\frac{d^m}{d\vartheta^m}\mathcal{I}_n(\boldsymbol{x})\Big\|_{L^\infty} \leq c_5 (n-1)^m \big\|\mathcal{I}_n(\boldsymbol{x})\big\|_{L^\infty}$, $\boldsymbol{x} \in \mathbb{R}^n$, $m \geq 1$ (inverse inequality)

Complexity Analysis

- Measure accuracy in terms of mean squared error:

$$\mathrm{MSE}\big(\hat{\Phi}_L^{(m)}\big) := \mathbb{E}\left(\left\|\hat{\Phi}_L^{(m)} - \Phi^{(m)}\right\|_{L^\infty}^2\right)\,, \quad m \in \mathbb{N}_0\,, \qquad L^\infty \equiv L^\infty(\Theta)\,.$$

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- **Notation:** $\mathrm{Var}(\xi) = \mathbb{E}\big(\|\xi - \mathbb{E}(\xi)\|_{\ell^\infty}^2\big)$, for any r.v. ξ with values in $\mathbb{R}^n$
- **Useful technical result**: let $\big(\xi^{(1)}, \dots, \xi^{(N)}\big) \subset \mathbb{R}^n$ independent, then

$$\mathrm{Var}\Big(\sum_{i=1}^{N} \xi^{(i)}\Big) \le c\ln(n)\sum_{i=1}^{N}\mathrm{Var}\big(\xi^{(i)}\big)$$

Theorem: A-priori MSE complexity ($h_{\ell-1}/h_\ell = s > 1$) for Spline interpolation [K., Nobile. 2018]

Suppose that:

- $\sup_{\vartheta\in\Theta}\left|\Phi(\vartheta) - \mathbb{E}\left(\phi(\vartheta, Q_l)\right)\right| = \mathcal{O}(h_l{}^\alpha)$
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Let $m = 0$. If $\Phi \in C^{k+1}(\Theta)$ and $2\alpha \geq \min\{\beta,\gamma\}$, then there exists an MLMC estimator $\hat{\Phi}_L$ of Φ such that $\mathrm{MSE}\left(\hat{\Phi}_L\right) = \mathcal{O}(\varepsilon^2)$ with

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- Similar complexity for Φ analytic and using an interpolation in global polynomials:

$$\text{comp. cost} \lesssim \varepsilon^{-2}|\ln(\varepsilon)|^4 + |\ln(\varepsilon)|^3 \begin{cases} \varepsilon^{-2}\,, & \beta > \gamma\,, \\ \varepsilon^{-2}\ln(\varepsilon)^2\,, & \beta = \gamma\,, \\ \varepsilon^{-\left(2+\frac{\gamma-\beta}{\alpha}\right)}|\ln(\varepsilon)|^{\frac{\gamma-\beta}{\alpha}}\,, & \beta < \gamma\,. \end{cases}$$

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If $\Phi \in C^{k+1}(\Theta)$ and $m \leq k$, then there exists $\hat{\Phi}_L$ such that $\mathrm{MSE}\left(\hat{\Phi}_L^{(m)}\right) = \mathcal{O}(\varepsilon^2)$:

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- Again, a similar complexity result is available for Φ analytic:

$$\text{comp. cost} \lesssim \varepsilon^{-2}|\ln(\varepsilon)|^{4(1+m)} + |\ln(\varepsilon)|^{3+4m} \begin{cases} \varepsilon^{-2} , & \beta > \gamma , \\ \varepsilon^{-2} \ln(\varepsilon)^2 , & \beta = \gamma , \\ \varepsilon^{-\left(2+\frac{\gamma-\beta}{\alpha}\right)} |\ln(\varepsilon)|^{\frac{\gamma-\beta}{\alpha}(1+2m)} , & \beta < \gamma . \end{cases}$$

MLMC for CDF, quantile, and CVaR: error control

- Recall that key quantities such as the CDF, quantile, and CVaR are all derived from the function

$$\Phi(\vartheta) = \mathbb{E}\big(\phi(\vartheta, Q)\big) \quad \text{for} \quad \phi(\vartheta, Q) = \vartheta + \frac{1}{1-\tau}(Q-\vartheta)^+ .$$

- However, complexity result with $m = 1$ applies directly only to Φ'. In fact, it “only” guarantees that $\mathrm{MSE}\big(\hat{\Phi}'\big) = \mathcal{O}(\varepsilon^2)$. What about these other derived quantities?

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Corollary

For $\tau \in (0, 1)$, let $\hat{\Phi}$ be the MLMC estimator of $\Phi \in C^{k+1}(\Theta)$, $k \geq 1 = m$, so that $\mathrm{MSE}\big(\hat{\Phi}'\big) = \mathcal{O}(\varepsilon^2)$. If the true τ-quantile is an interior point of Θ, then

$$\max\{\text{quantile MSE}, \text{CVaR MSE}, \text{unif. CDF MSE}\} = \mathcal{O}(\varepsilon^2) \ ,$$

at a cost dominated by $\mathrm{cost}\big(\hat{\Phi}'\big) = \mathrm{cost}(\text{CDF est.})$.

Corollary provides an all-at-once approach for the simultaneous approximation of CDF, quantiles, and CVaR.

Toy example: the characteristic function

- Let's consider the *toy model* to describe a European call option again, i.e., asset follows

$$dS = rS\,dt + \sigma S\,dW \ , \quad S(0) = S_0 \ ,$$

- Quantity of interest is the discounted "payoff": $Q := e^{-rT}\max\big(S(T) - K, 0\big)$

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- Consequently, its CDF $F_Q := \mathbb{E}\big(I(Q \leq \cdot)\big)$ has a jump discontinuity at the origin:

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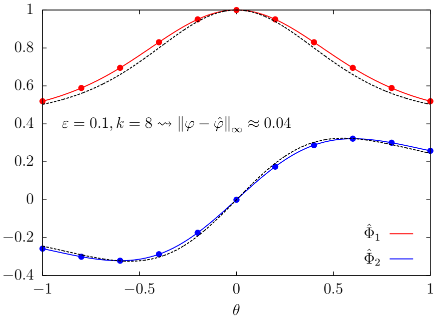
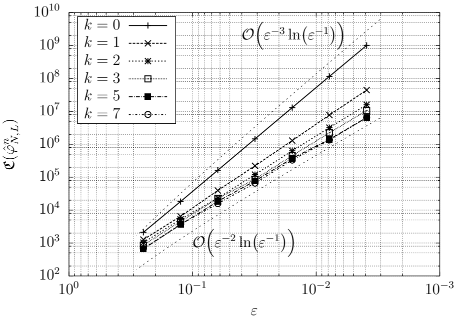
  $\rightsquigarrow$ we **cannot** guarantee a uniformly accurate MLMC CDF-approximation, if $0 \in \Theta$.
- But we **can** guarantee an accurate MLMC approximation of the **characteristic function** to characterize probability distribution of Q:

$$\varphi_Q(\vartheta) = \mathbb{E}\big(\underbrace{\cos(\vartheta Q)}_{=:\phi_1(\vartheta,Q)}\big) + \mathrm{i}\,\mathbb{E}\big(\underbrace{\sin(\vartheta Q)}_{=:\phi_2(\vartheta,Q)}\big) \equiv \Phi_1(\vartheta) + \mathrm{i}\,\Phi_2(\vartheta)\ ,$$

- **NB:** functions ϕ_i are smooth, no derivatives required (i.e., $m = 0$), moment approximations via post-processing possible [K., Nobile. 2018].

- Milstein scheme with $h_\ell = 2^{-\ell}T$; $\Theta = [-1, 1]$, $r = \frac{1}{20}$, $\sigma = \frac{1}{5}$, $T = 1$, $K = 10 = S_0$.





PDE toy example

Consider a simple Poisson equation

$$-\Delta u = f \,, \quad \text{in } D = (0,1)^2 \,,$$

with homogeneous Dirichlet boundary conditions and random forcing term f given by $f(x) = -72\xi({x_1}^2 + {x_2}^2 - x_1 - x_2)$, $\xi \sim \chi_1^2$. **Quantity of interest:** $Q := \int_D u \, dx = \xi$.

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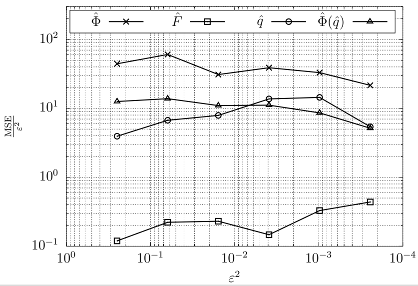
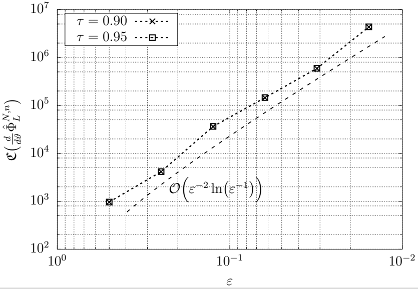
We approximate $\Phi(\vartheta) = \vartheta + \frac{1}{1-\tau}\mathbb{E}\left[(Q-\vartheta)^+\right]$ with $\tau = 0.95$ on $\vartheta \in \Theta = [0,10]$ for $k = 3 \rightsquigarrow$ theory predicts cost $= \mathcal{O}\left(\varepsilon^{-2.3}\ln(\varepsilon^{-1})\right)$, appears to be conservative.





Practical bottleneck: a-priori bounds

Asymptotic complexity analysis is based on **a-priori upper bounds** for error estimates:

$$\frac{\mathrm{MSE}\big(\hat{\Phi}_L^{(m)}\big)}{3} \leq \|\Phi^{(m)} - \mathcal{I}_n^{(m)}(\Phi)\|_{L^\infty}^2 + \|\mathcal{I}_n^{(m)}(\Phi - \Phi_L)\|_{L^\infty}^2 + \mathbb{E}\big(\|\mathcal{I}_n^{(m)}(\Phi_L - \Phi_L^{\mathtt{MLMC}})\|_{L^\infty}^2\big)$$

$$\leq C_1(m)^2 n^{-2(k+1-m)} + C_2(m)^2(n-1)^{2m} {b_L}^2 + C_2(m)^2(n-1)^{2m}\log(n)\sum_{\ell=0}^{L}\frac{V_\ell}{N_\ell}$$

where $b_L = \|\Phi(\boldsymbol{\theta}) - \Phi_L(\boldsymbol{\theta})\|_{\ell^\infty}$ and $V_\ell = \mathrm{Var}\big(\phi(\boldsymbol{\theta}, Q_\ell) - \phi(\boldsymbol{\theta}, Q_{\ell-1})\big)$.

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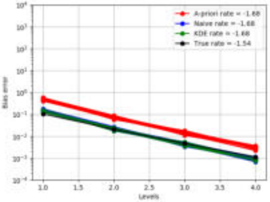
In particular, the *inverse inequality* makes error bound **severely conservative**. For example, the bias decay:



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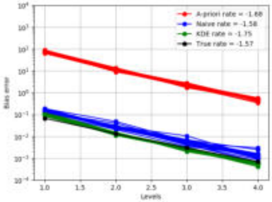

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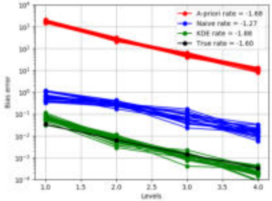

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Refined a-posterior error estimators I: bias

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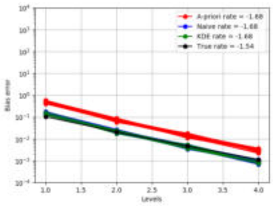
A-posteriori bias error estimation:



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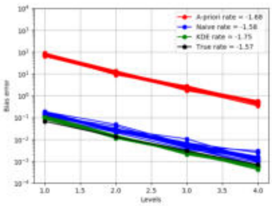

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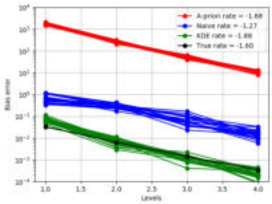


$m = 2$

Refined a-posterior error estimators II: statistical error

Idea: Statistical error estimator via bootstrapping MLMC estimators:

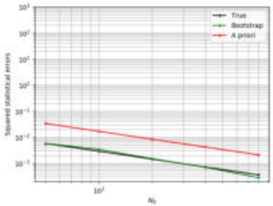
- "Observation": an MLMC estimator is defined through the hierarchy of samples

$$\left\{ \left\{ Q_\ell^{(i,\ell)}, Q_{\ell-1}^{(i,\ell)} \right\}_{i=1}^{N_\ell} \right\}_{\ell=0}^{L}$$

- Bootstrap principle: resample $N_{\mathrm{bs}} \gg 1$ "new" MLMC estimators Ψ_j

$$\mathbb{E}\left(\left\| \mathcal{I}_n^{(m)} (\Phi_L - \Phi_L^{\mathrm{MLMC}}) \right\|_{L^\infty}^2 \right) \approx \frac{1}{N_{\mathrm{bs}}} \sum_{j=1}^{N_{\mathrm{bs}}} \left\| \mathcal{I}_n^{(m)} \left(\Psi_j(\theta) - \bar{\Psi}(\theta) \right\|_{L^\infty}^2$$

A-posteriori statistical error estimation:



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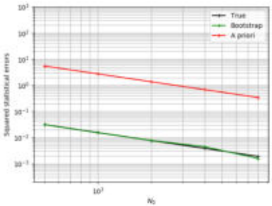

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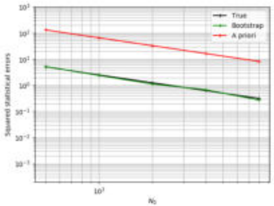

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Implication for an (adaptive) MLMC implementation

- A-priori bounds yield error estimator that is convenient for an (adaptive) implementation:

$$\mathbb{E}\left(\left\|\mathcal{I}_n^{(m)}(\Phi_L - \Phi_L^{\mathtt{MLMC}})\right\|_{L^\infty}^2\right) \leq K(n,m)\sum_{\ell=0}^{L}\frac{V_\ell}{N_\ell} \approx K(n,m)\sum_{\ell=0}^{L}\frac{\hat{V}_\ell}{N_\ell}$$

- The a-posterior error bound does not provide level-wise errors

$$\mathbb{E}\left(\left\|\mathcal{I}_n^{(m)}(\Phi_L - \Phi_L^{\mathtt{MLMC}})\right\|_{L^\infty}^2\right) \approx \frac{1}{N_{\text{bs}}}\sum_{j=1}^{N_{\text{bs}}}\left\|\mathcal{I}_n^{(m)}\left(\Psi_j(\boldsymbol{\theta}) - \bar{\Psi}(\boldsymbol{\theta})\right)\right\|_{L^\infty}^2 =: \left(\hat{e}_{\text{a-post}}^{(m)}\right)^2$$

- Rescaling idea: we aim for redefined levelwise variances $\tilde{V}_\ell$ such that $\left(\hat{e}_{\text{a-post}}^{(m)}\right)^2 = \sum_{\ell=0}^{L}\frac{\tilde{V}_\ell}{N_\ell}$

Lemma

There exist positive constants $\mathfrak{K}_1(n)$ and $\mathfrak{K}_2(n)$: $\mathfrak{K}_1(n)\sum_{\ell=0}^{L}\frac{V_\ell}{N_\ell} \leq \sum_{m=0}^{2} k_m\left(\hat{e}_{\text{exact}}^{(m)}\right)^2 \leq \mathfrak{K}_2(n)\sum_{\ell=0}^{L}\frac{V_\ell}{N_\ell}$

- Rescaling heuristic: $\tilde{V}_\ell = r_e\hat{V}_\ell\ , \quad r_e = \dfrac{\sum_{m=0}^{2} k_m\left(\hat{e}_{\text{a-post}}^{(m)}\right)^2}{\sum_{k=0}^{L}\hat{V}_k/N_k}$

Computational Example

In [Ayoul-Guilmarda, Ganesh, K., Nobile. 2023] we combine these with an adaptive, *continuation MLMC framework* for an efficient implementation (Python library `XMC`) that can be tuned to target CDF, VaR, or CVaR.

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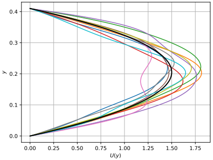
Example: (steady incompressible) Navier-Stokes Flow over a Cylinder in a Channel



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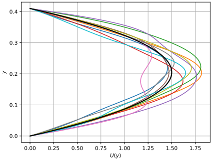

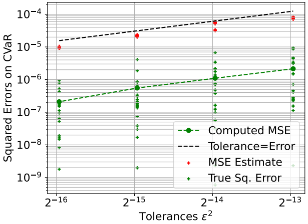

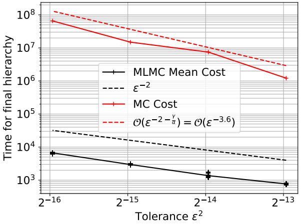

Take home message:

- we introduced a uniformly accurate MLMC estimator for a parametric expectation and its derivatives
- the approach enables approximating the characteristic function as well as CDF, PDF, VaR, and CVaR
- refined, a-posterior error estimates are required for computationally “heavy” problems when derivatives are required.

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Thank you for your attention.

Further details:

SK and F. Nobile. Multilevel Monte Carlo Approximation of Functions. *SIAM/ASA J. Uncertain. Quantif.*, **3**(6):1256–1293, 2018.

Q. Ayoul-Guilmard, S. Ganesh, SK, and F. Nobile. Quantifying uncertain system outputs via the multi-level Monte Carlo method – distribution and robustness measures. *Int. J. Uncertain. Quantif.*, **13**(5):61–98, 2023.