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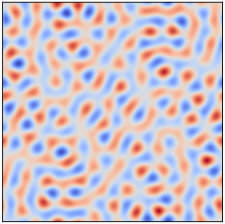

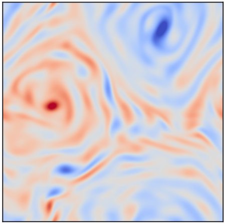

Downscaling Data Assimilation Techniques Applied to Low Froude Number Shallow Water Flows

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- dynamical downscaling becoming popular, easy to implement
- difficulties, e.g. gridded nudging discrete in nature
- Titi and co-workers: rigorous mathematical framework for downscaling and assimilation techniques

Aim:

- extend developments, design new efficient algoritms for extreme-scale calculations
- computationally test efficiency and validity for practical situations

Data Assimilation by Interpolant Observables

Evolution governed by general dissipative dynamical system

$$\frac{du}{dt} = \mathcal{F}(u)$$

where initial data u_0 is missing. Find algorithm for constructing $v(t)$ from observational measurements $I_h(u(t))$.

Nudging:

$$\frac{dv}{dt} = \mathcal{F}(v) - \mu I_h(v) + \mu I_h(u), \quad \mu: \text{relaxation parameter}$$

$$v(0) = v_0$$

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Spectrally-filtered discrete-in-time downscaling:

$$\begin{aligned} v_0 &= Ju(t_0) \\ v^{n+1} &= ES(t_{n+1}, t_n; v_n) + Ju(t_{n+1}) \\ v(t) &= S(t, t_n; v_n) \quad \text{for } t \in [t_n, t_{n+1}) \end{aligned}$$

Construct approximate solution $v(t)$, which converges to $u(t)$ over time.

Let $\Omega = [0, 2\pi]^2$ be the 2π-periodic torus, find $\boldsymbol{u}$, s.th.

$$\begin{aligned} \boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} \\ \nabla\cdot\boldsymbol{u} &= 0 \end{aligned}$$

$\boldsymbol{u}$: fluid velocity, p: pressure
$\boldsymbol{f}$: body force applied to the fluid
ν: kinematic viscosity

Azouani et al., 2014:
Let $\Omega = [0, L]^2$ and let $\boldsymbol{u}$ be a solution to the NSE with periodic BC. If I_h satisfies the inequality

$$\|\boldsymbol{u} - I_h(\boldsymbol{u})\|_{L^2}^2 \leq \gamma_1 h^2 \|\boldsymbol{u}\|_{H^1}^2 + \gamma_2 h^4 \|\boldsymbol{u}\|_{H^2}^2$$

and

$$\frac{L^2}{h^2} \geq \frac{c_0 L^2 \mu}{\nu} \geq c_2 Gr(1 + \log(1 + Gr)), \quad c_0 = c_0(\gamma_1, \gamma_2), c_2 = \text{const.}$$

then $\|\boldsymbol{v}(t) - \boldsymbol{u}(t)\| \to 0$ as $t \to \infty$.

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Interpolant Observables

- Orthogonal projection onto low Fourier modes:

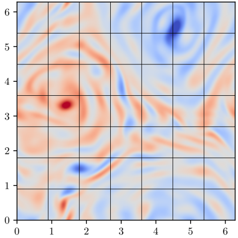
$$I_h(u)(x) = P_\lambda(u)(x) = \sum_{|k|\leq\lambda} \hat{u}_k \, e^{ik\cdot x}$$

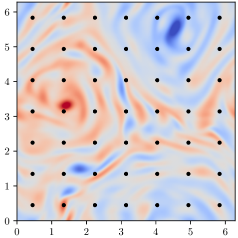
- Interpolation from volume elements

$$I_h(u)(x) = \sum_{j=1}^{N} \bar{u}_j \, \chi_{Q_j}(x)$$

- Interpolation from nodal measurements

$$I_h(u)(x) = \sum_{j=1}^{N} u(x_j) \left(\chi_j(x) - \frac{1}{|\Omega|} \int_\Omega \chi_j \right)$$





The Zero Froude Number SWE

Shallow water equations:

$$
\begin{aligned}
h_t + \nabla \cdot (h\mathbf{v}) &= 0 \\
(h\mathbf{v})_t + \nabla \cdot (h\mathbf{v} \circ \mathbf{v}) + \frac{1}{\mathrm{Fr}^2} h \nabla h &= 0
\end{aligned}
$$

Zero Froude number limit ($\mathrm{Fr} \to 0$), rigid lid approximation:

$$
\begin{aligned}
h_t \quad + \quad \nabla \cdot (h\mathbf{v}) \qquad\qquad\qquad &= \quad 0 \\
(h\mathbf{v})_t \quad + \quad \nabla \cdot (h\mathbf{v} \circ \mathbf{v}) \quad + \quad h \nabla h^{(2)} &= \quad \mathbf{0}
\end{aligned}
$$

- $h = h_0(t)$ given through boundary conditions.
- mass conservation becomes divergence constraint for velocity field:

$$
\int_{\partial V} h\mathbf{v} \cdot \mathbf{n} \, d\sigma = -|V| \frac{dh_0}{dt} \qquad \text{for } V \subset \Omega
$$

- $h^{(2)}$: second order height perturbation; Lagrange multiplier

Task: Construct a scheme, which . . .

- . . . allows time steps independent of the Froude number
- . . . conserves mass, momentum, total energy:

$$\mathbf{U}_C^{n+1} = \mathbf{U}_C^n - \frac{\Delta t}{|C|} \sum_{I \in \mathcal{I}_{\partial C}} |I| \, \mathbf{F}_I$$

- . . . is second order accurate in time and space
- . . . uses machinery of Godunov-type methods
- . . . requires the solution of at most linear, scalar equations
- . . . ensures, that for $\mathrm{Fr} = 0$, advection velocities and final momentum satisfy divergence constraint

Zero Froude number SWE:

$$\nabla \cdot (h_0 \boldsymbol{v}) = -\frac{dh_0}{dt}$$

$$(h_0 \boldsymbol{v})_t + \nabla \cdot (h_0 \boldsymbol{v} \circ \boldsymbol{v}) + h_0 \nabla h^{(2)} = \mathbf{0}$$

Method in conservation form:

$$\mathbf{U}_C^{n+1} = \mathbf{U}_C^n - \frac{\Delta t}{|C|} \sum_{I \in \mathcal{I}_{\partial C}} |I| \, \mathbf{F}_I^{n+1/2}$$

$$\mathbf{F}_I^{n+1/2} := \mathbf{F}_I^* + \mathbf{F}_I^{\mathrm{MAC}} + \mathbf{F}_I^{\mathrm{P2}}$$

- advective fluxes $\mathbf{F}_I^*$ from second order Godunov-type method
- $\mathbf{F}_I^{\mathrm{MAC}}$ from (MAC) projection, corrects advection velocity divergence
- $\mathbf{F}_I^{\mathrm{P2}}$ from second projection, which adjusts new time level divergence of cell-centered velocities

mimic Zero Froude Number results by decomposing height into:

$$h(t,\boldsymbol{x};\mathrm{Fr}) = h_0(t,\boldsymbol{x}) + \mathrm{Fr}^2\, h'(t,\boldsymbol{x})$$
$$h_0(t,\boldsymbol{x}) = H_0(t) - b(t,\boldsymbol{x})$$

$$h_t + \nabla\cdot(h\boldsymbol{v}) = 0$$
$$(h\boldsymbol{v})_t + \nabla\cdot(h\boldsymbol{v}\circ\boldsymbol{v}) + h\nabla h' = 0$$

Extension of (zero Froude number) projection method by inclusion of local time derivatives of h' into projection steps

1. explicit predictor for advective fluxes
2. first correction (advection velocity components)
3. **second correction** (pressure term)

Diffusion Term

- Add as source term into explicit predictor
- Split-implicit method, using Strang Splitting:

$$(h\boldsymbol{v})^* = P_{\text{adv}}^{\Delta t/2}\left(P_{\text{diff}}^{\Delta t}\left(P_{\text{adv}}^{\Delta t/2}(h\boldsymbol{v})^n\right)\right)$$

Crank-Nicholson Scheme for diffusion step:

$$(h\boldsymbol{v})^{n+1} = (h\boldsymbol{v})^n + \frac{\Delta t}{2}\left[\nabla\cdot\nu h\nabla(\boldsymbol{v}^{n+1}) + \nabla\cdot\nu h\nabla(\boldsymbol{v}^n)\right]$$

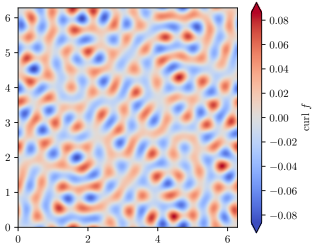
Solution of two Helmholtz problems for the velocity components

Body Force f

- Let $f = \gamma f_0$ being a time-independent forcing function.
- f_0 supported on an annulus of Fourier modes, such that $10 \le |k| \le 12$
- f_0 specified in vorticity representation in terms of $g_0 = \nabla \times f_0$ with

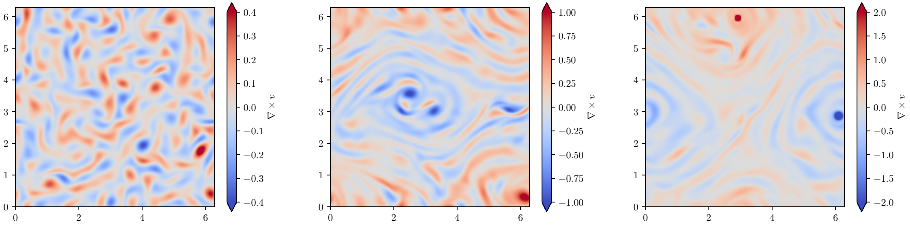
$$\hat{f}_k = \frac{\hat{g}_k}{k_1^2 + k_2^2} \begin{pmatrix} -ik_2 \\ ik_1 \end{pmatrix}, \quad g = \sum_{10 \le |k| \le 12} \hat{g}_k \phi_k$$

  with $\hat{f}_k = \overline{\hat{f}_{-k}}$ and $k \cdot \hat{f}_k = 0$
- Fourier coefficients $\hat{g}_k$ Gaussian distributed (see Olson and Titi, 2008)
- discrete f_h further projected into space of discretely divergence-free functions.



Resulting Flow, $\nu = 0.0001$

Depending on scaling γ, different flow regimes:

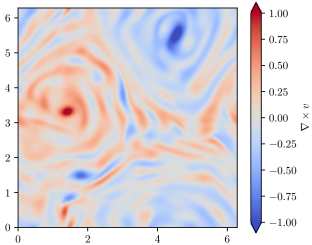


$Gr(f) = 175\,000$ $\quad$ $Gr(f) = 1\,500\,000$ $\quad$ $Gr(f) = 2\,500\,000$

$Gr(f) = \frac{1}{\nu^2} \|f\|_{L^2}$ (Grashof number)

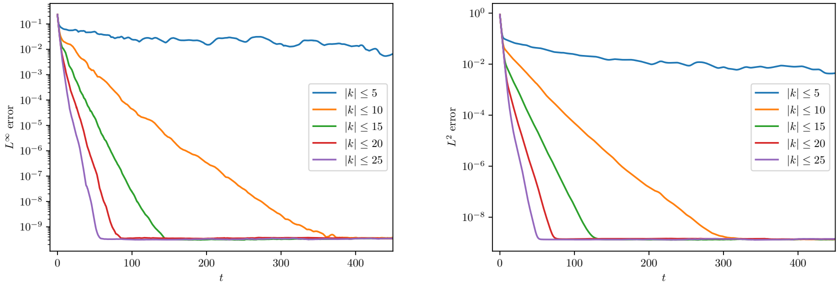
Data Assimilation Experiments

- forcing function f with $Gr(f) = 1\,500\,000$
- viscosity $\nu = 0.0001$
- $\Omega = [0, 2\pi]^2$ with 256^2 grid cells
- CFL number 0.8 (determined by explicit predictor)
- zero velocity field at $t = -6600$, evolve until $t = 0$ for initial condition
- DA by nudging with projection onto low Fourier modes: $\lambda = 5, 10, 15, 20, 25$



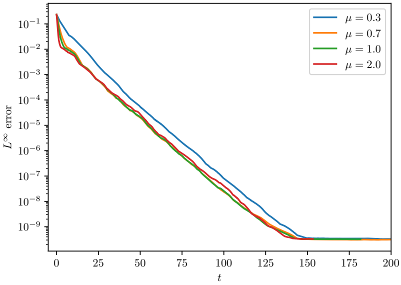
initial velocity field (reference solution)

Error with respect to reference solution ($\mu = 0.7$):



$\sim$ 80 modes needed to experimentally obtain convergence in time!
(see also Olson and Titi (2008), Gesho et al. (2015), ...)

Convergence wrt μ:



Summary

- data assimilation by interpolant observables
- finite volume projection method for zero Froude number flows
- semi-implicit extention for low Froude number SWE
- DA with spectral nudging

- Outlook
  - different interpolant observables, downscaling
  - modeling of atmospheric flows with moisture, stochastic models
  - effective methods to treat ensemble of system states