A consistent framework for stochastic representation of large-scale geophysical flows

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Workshop on Conservation Principles, Data, and Uncertainty in Atmosphere-Ocean Modelling
**Introduction**

**Geophysical flow analysis**

- Strong interest on the use of stochastic filters and ensemble methods for data assimilation and forecasting

- Particularly interesting to combine a partially known evolution law with noisy data

⇒ Require stochastic version of the evolution law and/or a modeling of the dynamics errors

Several methodological framework proposed in the literature (Berner et al. 2017, Franzke et al. 2015)

- additive/multiplicative forcing (Buiza et al. 99), backscattering (Leight 71), (Mason and Thomson 92)

- Low/fast modes decomposition (Majda et al. 99, Franzke et al. 05), scale separation (Grooms and Majda).

- Approaches based on stochastic transport (Holm 15, Mémin 14)

Explore expressions of dynamical models under location uncertainties
Location uncertainties

Principle

- Fluid particles displacement can be separated in two components: a smooth differentiable drift $\mathbf{w}$ and a random uncertainty function $\sigma \mathrm{d}\mathbf{B}_t$

- Flow:
  \[
  \mathbf{X}_t = \mathbf{X}_{t_0} + \int_{t_0}^{t} \mathbf{w}(\mathbf{X}_s, s) \mathrm{d}s + \int_{t_0}^{t} \sigma(\mathbf{X}_s, s) \mathrm{d}\mathbf{B}_s,
  \]

- Displacement:
  \[
  \mathrm{d}\mathbf{X}_t = \mathbf{w}(\mathbf{X}_t, t) \mathrm{d}t + \sigma(\mathbf{X}_t, t) \mathrm{d}\mathbf{B}_t, \quad \text{with } \mathbf{X}_{t_0} = \mathbf{x},
  \]

- Eulerian description of the velocity fields:
  \[
  \mathbf{U}(\mathbf{x}, t) = \mathbf{w}(\mathbf{x}, t) + \sigma(\mathbf{x}, t) \dot{\mathbf{B}}_t.
  \]

- $\mathbf{U}$ should be solution of Navier Stokes equation derived from Newton 2nd law
Noise term

Uncertainty random field

- **B**: Brownian motion function
- **Uncertainty**: Diffusion tensor and White noise on Ω
  \[(σ(x, t)dB_t)^i = \sum_j \int_Ω \tilde{σ}^{ij}(x, y, t)dB^j_t(y)dy.\]
  - Diffusion **σ**
    - Hilbert-Schmidt operator (Covariance of finite norm)
  - Covariance of the turbulent component
    \[Q_{ij}(x, y, t, t') = \sum_k \int_Ω \tilde{σ}^{ik}(x, y', t) \tilde{σ}^{jk}(y', y, t)dy'δ(t − t')dt\]
    \[Q_{ij}(x, x, t, t') \triangleq a_{ij}(x)δ(t − t')dt\]
### Example: Kraichnan turbulence model

- Homogeneous diffusion provides homogeneous covariance tensor with spatially constant variance
- Incompressible fluid $d\xi_t^\zeta = \mathcal{P} \ast dB_t^\zeta$
- Spectral correlation defined as

\[
\hat{Q}(k)_{ij} = |k|^{-\zeta-d} \left( \delta_{ij} - \frac{k_i k_j}{|k|^2} \right) (\hat{\psi}_\zeta^\gamma)^2 dt
\]

- Quadratic variation process for a passband spectral cutoff ($1_{[\kappa,\gamma]}(k)$)

\[
a_{ij} = C_{\zeta,d} \zeta^{-1} (L^\zeta - \ell_D^\zeta) \delta_{ij}
\]
Stochastic Reynolds transport theorem

Volumetric rate of change

Volumetric rate of change of a scalar process \( q(x, t) \) transported by

\[
dX_t = w(X_t, t)dt + \sigma(X_t, t)dB_t, \text{ with } \nabla \cdot \sigma = 0
\]

\[
d \int_{V(t)} q(x, t)dx = \int_{V(t)} \left( dtq + \nabla \cdot \left( qdX_t - \frac{1}{2} \nabla \cdot (aq)^T dt \right) \right) dx
\]

Example for the smooth Kraichnan model:

\[
d \int_{V(t)} q(x, t)dx = \int_{V(t)} \left[ dtq + (\nabla \cdot (qw) - \frac{1}{2} \gamma \Delta q)dt + \nabla q^T d\xi_t^c \right] dx,
\]
Stochastic Reynolds transport theorem

**Conservation of extensive scalar**

**Conservation constraint on the transported volume:**
\[ d \int_{V(t)} q(x, t) dx = 0 \]

**Evolution law:**
\[ d_t q + \nabla q \cdot (\tilde{w} dt + \sigma dB_t) - \nabla \cdot \left( \frac{1}{2} a \nabla q \right) dt = q \nabla \cdot \tilde{w} dt, \]
\[ D_t q = q \nabla \cdot \tilde{w} dt \]

**Effective drift:**
\[ \tilde{w} = w - \frac{1}{2} (\nabla \cdot a) \]

**Dissipative operator:**
\[ \int_{\Omega} q \nabla \cdot (a \nabla q) = -\sum_{i,j} \int_{\Omega} \partial_i q \ a^{ij} \partial_j q \leq 0 \]

**Incompressibility:** A constant value of the scalar implies
\[ \nabla \cdot (\sigma dB_t) = 0, \ \nabla \cdot \tilde{w} = 0 \]
Energy conservation of transported scalar (incompressible flow)

\[ \int_{\Omega} dt (q^2) = 0, \]

Decomposition: \( q = \mathbb{E}(q|\tilde{w}) + (q - \mathbb{E}(q|\tilde{w})) \)

Energy conservation:

\[ \frac{d}{dt} \mathbb{E} \|q\|_{\mathcal{L}^2(\Omega)}^2 = \frac{d}{dt} \|\mathbb{E}(q)\|_{\mathcal{L}^2(\Omega)}^2 + \frac{d}{dt} \int_{\Omega} \text{Var}(q) dx = 0 \]

Mean field energy:

\[ \partial_t \int_{\Omega} (\mathbb{E}(q))^2 dx = -\frac{1}{2} \int_{\Omega} \nabla \mathbb{E}(q) a \nabla \mathbb{E}(q) dx \leq 0. \]

Mean field energy decreases, Variance increases by the same amount.
Conservation of momentum

Considering stochastic conservation principle (in a distribution sense)

\[ D_t \left( \rho \left( w(x, t) + \sigma(x, t) \frac{dB_t}{dt} \right) \right) = F(x, t) \]

⇒ General (incompressible) stochastic Navier-Stokes equations

\[ D_t (\rho w) = -\nabla (\rho dt + dp_t') + \mu \Delta (wdt + \sigma dB_t) \]

\[ D_t \rho = 0 \]

\[ \nabla \cdot (\sigma dB_t) = 0, \quad \nabla \cdot w - \nabla \cdot (\nabla \cdot a) = 0 \]

incompressibility

with

\[ D_t q = dt + \nabla q \cdot (\tilde{w} dt + \sigma dB_t) - \frac{1}{2} \nabla \cdot (a \nabla q) dt \]
LES under location uncertainty

- Large scale drift of finite variation $\Rightarrow$ separation of Brownian terms and finite variation terms

**Momentum equations**

$$\partial_t \mathbf{w} + (\tilde{\mathbf{w}} \cdot \nabla) \mathbf{w} - \frac{1}{2} \nabla \cdot (\mathbf{\tau}) = -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{w},$$

**Effective drift**

$$\tilde{\mathbf{w}} = (\mathbf{w} - \frac{1}{2}(\nabla \cdot \mathbf{a})^T)$$

**Subgrid tensor**

$$\mathbf{\tau} = ((a \nabla) \mathbf{w})$$

**Pressure random contribution**

$$\frac{1}{\rho} \nabla d\hat{p}_t = -(\sigma d\mathbf{B}_t \cdot \nabla) \mathbf{w} + \nu \Delta (\sigma d\mathbf{B}_t),$$

**Mass conservation**

$$\nabla \cdot (\sigma d\mathbf{B}_t) = 0, \ \nabla \cdot \mathbf{w} - \frac{1}{2} \nabla \cdot \nabla \cdot \mathbf{a} = 0$$
Incompressible case

**Subgrid model:**

- Constant eddy viscosity model for Kraichnan uncertainty model
- Smagorinsky model for $a = c \| \mathbf{S} \| \mathbb{I}$ with $\| \mathbf{S} \|$ smooth enough
- Noise on isopycnal surfaces $\Rightarrow$ Gent-McWilliams
- New LES schemes with $a$ defined from local variance
- Stochastic models from POD noise on high resolution simulations, resolved fluctuations, predefined covariances, targeted dissipation (hyperviscosity), etc.

**Simulations:** Wake flows, channel flows, Green-Taylor flow, geophysical flow approximations
Geophysical flow modelling

- Same derivation and assumption as in the deterministic case
- Stratification, traditional approximation, etc.
- Introduction of the transport stochastic operator (material derivative)
- Noise specification for the numerical simulations
Geophysical flow modeling

Simple Boussinesq equations (LU)

**Momentum equations**

\[ \mathbb{D}_t w + f k \times (u + (\sigma dB_t)_H) = b k - \frac{1}{\rho_b} \nabla p' + \mathcal{F}(w), \]

**Effective drift**

\[ \tilde{w} = \begin{pmatrix} \tilde{u} \\ \tilde{w} \end{pmatrix} = w - \frac{1}{2} (\nabla \cdot a)^T, \]

**Buoyancy equation**

\[ \mathbb{D}_t b + N^2 (\tilde{w} dt + (\sigma dB_t)_z) = \frac{1}{2} \nabla \cdot (a \cdot z N^2) \, dt, \]

\[ b = -\frac{g}{\rho_b} \rho', \quad N^2(z) = -\frac{g}{\rho_b} \partial_z \rho_0(z) \]

**Incompressibility**

\[ \nabla \cdot w - \frac{1}{2} \nabla \cdot \nabla \cdot a = 0; \quad \nabla \cdot (\sigma dB_t) = 0 \]
<table>
<thead>
<tr>
<th>Stochastic approximated model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zoology of approximated models obtained from power series expansion in small number (Rossby) and scaling</td>
</tr>
<tr>
<td>PG, QG, SQG, ...</td>
</tr>
<tr>
<td>Same strategy and Proper scaling of the noise term</td>
</tr>
<tr>
<td>⇒ Stochastic PG, QG, SQG, ... see talks of Long Li and Valentin Resseguier</td>
</tr>
<tr>
<td>Global energy conservation, Modified enstrophy conservation for general noises</td>
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<tr>
<td>Structure preservation</td>
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</table>
Usefulness of physically consistent Stochastic modeling

- Accuracy of large-scale models and error representation
- Fast exploration of attractor’s regions
- New physical modeling capabilities
Geophysical flow modelling

Usefulness of physically consistent Stochastic modeling

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Rayleigh-Bénard convection (incompressible fluid with $T_b > T_u$)

- Galerkin projection on Fourier modes of Navier-Stokes with Boussinesq approximation

\[
\frac{dX}{dt} = Pr(Y - X) - \frac{4}{2\gamma} X
\]

\[
dY = [X(\rho - Z) - Y - \frac{4}{2\gamma} Y]dt + \frac{\rho - Z}{\gamma^{1/2}} dB_t
\]

\[
dZ = [XY - bZ - \frac{8}{2\gamma} Z]dt + \frac{Y}{\gamma^{1/2}} dB_t
\]

**Usual model** $\sim$ DNS

**Diffusive model** $\sim$ LES

**Model under uncertainty**
Geophysical flow modelling

Attractor visits example (strong noise and low noise)

Points of the Lorenz attractor visited (ensemble of 100 particles)
LZ (a), LZD (b), SLZ (c); colors: time of the visit; $\gamma = 10; \gamma = 100$
Geophysical flow modelling

Empirical PDF of $X$ and spectrum of $Z$ strong noise $\gamma = 10$

![Graphs showing empirical PDF of X and spectrum of Z.]

10,000 realizations $t \in [0, 100]$

BSLZ: ad hoc stochastic Lorenz with multiplicative noise (Chekroun et al)

\[
\frac{dX}{dt} = Pr(Y - X)
\]

\[
dY = [X(\rho - Z) - Y]dt + \frac{Y}{\gamma^{1/2}} dB_t
\]

\[
dZ = [XY - bZ]dt + \frac{Z}{\gamma^{1/2}} dB_t
\]
Geophysical flow modelling

Empirical PDF of $X$ and spectrum of $Z$ weak noise $\Upsilon = 100$

10,000 realizations $t \in [0, 100]$

BSLZ: adhoc stochastic Lorenz with multiplicative noise (Chekroun et al)

\[
\frac{dX}{dt} = P_r(Y - X)
\]

\[
dY = [X(\rho - Z) - Y]dt + \frac{Y}{\Upsilon^{1/2}} dB_t
\]

\[
dZ = [XY - bZ]dt + \frac{Z}{\Upsilon^{1/2}} dB_t
\]
Lorenz attractor visit rate $\Upsilon = 10$ and $\Upsilon = 100$

$\#\{\text{boxes covering the attractor visited over } [0,T]\}/N$ (Computed on 10,000 realizations; red: SLZ, green: LZD, black: LZ)
Geophysical flow modelling

Usefulness of physically consistent Stochastic modeling

- Accuracy of large-scale models and error representation
- Fast exploration of attractor’s regions
- New physical modeling capabilities
Modified wall law expression in wall bounded flows

### Boundary layers and wall law

- Profile of an idealized mean velocity profile $u(z)$ over a plane wall with constant pressure

### Boundary layers structure

- Viscous layer (dominated by molecular friction)
- Turbulent layer (dominated by large-scale shear stress associated to the unresolved fluctuations)

Two different dynamical regimes piloted by different physics

**Buffer zone** at the interface
Modified wall law expression in wall bounded flows

Viscous layer stationary equations

**Large-scale component**: \( \nu \nabla^2 u = 0 \Rightarrow \partial_z u = C_1, \)

**Small scale component**: \( \nu \nabla^2 \sigma dB_t = 0 \Rightarrow \nabla^2 \sigma^{ij} = 0, \)

**Turbulent pressure**: \( dp_t = C_2, \)

**Incompressibility**: \( \nabla \cdot (\sigma dB_t) = 0 \)

\( \Rightarrow \) **Mean Velocity profile**

\[ \forall z \in [0, z_0] \quad u(z) \Delta t = \frac{1}{\nu} \bar{U}^2 z \delta u \Delta t + \epsilon z (\Delta t)^{1/2} \eta. \]

Small-scale component: 2D divergence free random field with variance, which depends on the wall shear stress variance and with a quadratic growth in \( z \).
Modified wall law expression in wall bounded flows

**Turbulent layer stationary equations**

Large-scale component: \(-\partial_z a_{zz} \partial_z u - \partial_z((a_{zz} + 2\nu)\partial_z u) = 0,\)

Turbulent pressure: \(\nabla_H dp_t = \partial_z u(\sigma dB_t)_z + \nu \nabla^2(\sigma dB_t)_H = 0,\)

Turbulent pressure: \(\partial_z dp_t = \nu \nabla^2(\sigma dB_t)_z,\)

Incompressibility: \(\nabla \cdot (\sigma dB_t) = 0\quad \nabla \cdot (\nabla \cdot a) = 0\)

Logarithmic law for the mean velocity profile if the turbophoresis drift is neglected and linear turbulent viscosity

**Buffer sublayer**

Incompressibility: \(\nabla \cdot \nabla \cdot a = 0 \implies \partial_{zz}^2 a = 0 \implies a_{zz}(z) = \tilde{\kappa} \tilde{U}_\tau (z - z_0)\)

At the interface \(z = z_0, \partial_z u(z_0) = \frac{1}{\nu} \tilde{U}_\tau^2 \delta u\)

\(\implies\) Mean velocity profile:

\[
\forall z \in [z_0, z_L]\quad u(z) = u(z_0) - \tilde{U}_\tau \frac{4\nu}{\kappa} \left( \frac{1}{\tilde{\kappa} \tilde{U}_\tau (z - z_0) + 2\nu} - \frac{1}{2\nu} \right) \delta u
\]
Modified wall law expression in wall bounded flows

Turbulent layer stationary equations

Large-scale component: \(-\partial_z a_{zz} \partial_z u - \partial_z ((a_{zz} + 2\nu) \partial_z u) = 0\),

Turbulent pressure: \(\nabla_H d\rho_t = \partial_z u (\sigma dB_t)_z + \nu \nabla^2 (\sigma dB_t)_H = 0\),

Turbulent pressure: \(\partial_z d\rho_t = \nu \nabla^2 (\sigma dB_t)_z\),

Incompressibility: \(\nabla \cdot (\sigma dB_t) = 0\) \(\nabla \cdot (\nabla \cdot a) = 0\)

Logarithmic sublayer

To get a logarithmic law, \(a_{zz} \sim \sqrt{z}\) and by continuity

\[a_{zz}(z) = \tilde{k} \tilde{U}_\tau (z_L - z_0) \sqrt{\frac{z}{z_L}}, \quad \forall z \in [z_L, z_1]\]

\[
\begin{align*}
  u(z) &= u(z_L) + \partial_z u(z_L) z_L \ln \left( \frac{z}{z_L} \right) \\
  \text{with the factor } z_L \partial_z u(z_L) &= \left(4\nu \tilde{U}_\tau^2 z_L\right) / \left(\tilde{k} \tilde{U}_\tau (z_L - z_0) + 2\nu\right)^2
\end{align*}
\]
Numerical results turbulent boundary layer flow

$$\tilde{u}^+$$ vs $$z^+$$

<table>
<thead>
<tr>
<th></th>
<th>$$Re_T$$</th>
<th>$$z_0^+$$</th>
<th>$$z_L^+$$</th>
<th>$$\tilde{\kappa}$$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1989</td>
<td>4.90</td>
<td>50.38</td>
<td>0.158</td>
</tr>
</tbody>
</table>
Geophysical flow modeling

Numerical results channel flow

<table>
<thead>
<tr>
<th>$Re_T$</th>
<th>$z_0^+$</th>
<th>$z_L^+$</th>
<th>$\tilde{\kappa}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5200</td>
<td>5.0</td>
<td>45.0</td>
<td>0.16</td>
</tr>
</tbody>
</table>
Role of small-scale inhomogeneity in flow structuration

From Euler-LU to Craik-Leibovich-LU

Euler-LU Momentum:

\[
d_t \mathbf{w} + ((\mathbf{w} \cdot \nabla) \mathbf{w} - \frac{1}{2} \nabla \cdot (a \nabla \mathbf{w}) \mathbf{d}t + 2 \Omega \times \mathbf{w} \mathbf{d}t = -\nabla p
\]

\[
\tilde{\mathbf{w}} = \mathbf{w} - \frac{1}{2} \nabla \cdot a = \mathbf{w} + \mathbf{w}_S
\]

Change of variable, introduction of a modified pressure, turbophoresis drift stationnary and quasi-harmonic:

\[
d_t \tilde{\mathbf{w}} + ((\tilde{\mathbf{w}} \cdot \nabla) \tilde{\mathbf{w}} - \frac{1}{2} \nabla \cdot (a(\tilde{\mathbf{w}} - \mathbf{w}_S)^T) + (\mathbf{w}_S \cdot \nabla)\sigma \mathbf{d}B_t = -\nabla \pi
\]

\[
- 2\Omega \times \tilde{\mathbf{w}} \mathbf{d}t + 2\Omega \times \mathbf{w}_S \mathbf{d}t - \mathbf{w}_S \times (\mathbf{w} \cdot \nabla) \sigma \mathbf{d}B_t
\]

LES/LU deterministic form corresponds to Craik-Leibovich system

\[
\partial_t \tilde{\mathbf{w}} + (\tilde{\mathbf{w}} \cdot \nabla) \tilde{\mathbf{w}} - \frac{1}{2} \nabla \cdot (a(\tilde{\mathbf{w}} - \mathbf{w}_S)^T) = -\nabla \pi - 2\Omega \times (\tilde{\mathbf{w}} - \mathbf{w}_S) - \mathbf{w}_S \times \tilde{\mathbf{w}}
\]

with pressure term \( \pi = p + \frac{1}{2} \| \mathbf{w} - \mathbf{w}_S \|^2 - \frac{1}{2} \| \tilde{\mathbf{w}} \|^2 \)
Role of small-scale inhomogeneity in flow structuration

Numerical simulation Chanel flow $Re_\tau = 590$

Simulation of a DNS, LES 1/48 (Dynamic Smagorinsky) and LU 1/48 (Full stochastic model with noise learned from POD/EOF on DNS)

row: DNS, LU, DSM; col: $y^+ = 50$, $y^+ = 100$, $y^+ = 200$
Role of small-scale inhomogeneity in flow structuration

Numerical simulation Chanel flow $Re_τ = 590$

Turbophoresis drift, Vorticity, Vortex force
Role of small-scale inhomogeneity in flow structuration

Numerical simulation Chanel flow $Re_T = 590$

1D FFT of vortex force components
Conclusion

Dynamics modeling through stochastic transport

- LU modeling allows a systematic derivation of stochastic fluid dynamics models
- Noise brings an additional degree of freedom for modeling
- Good representation of large-scale models (more accurate and faster)
- Data analysis tool (to decipher the role of small-scale) (see also Resseguier et al JFM 2017)
- Modeling tool to explain and simulate events related to small-scale activity
Perspectives

Ocean modelling

- Derivation of Air-Sea interaction models
- Generalize the idea of modified advection and instabilities
- Study of the wave solutions (Rossby waves, internal waves, ...)
- Theoretical study of the associated SPDEs
- Interaction of waves and mean large scale current
- Full stochastic simulation of realistic ocean dynamics models
- Data assimilation going toward non Gaussian models
Recent Publications

B. Pinier, E. Mémin, S. Laizet, R. Lewandowski
A stochastic flow model to predict the mean velocity in wall bounded flows, hal-inria, 2019

Y. Yang, E. Mémin
Estimation of physical parameters under location uncertainty using an Ensemble$^2$-Expectation-Maximization algorithm, QJRMS, 2019

B. Chapron, P. Dérian, E. Mémin, V. Resseguier
Large scale flows under location uncertainty: a consistent stochastic framework, QJRMS, 2018

P. Chandramouli, D. Heitz, S. Laizet, E. Mémin
Coarse large-eddy simulations in a transitional wake flow with flow models under location uncertainty, Comp. and Fluids. 2018

S. Kadri, Mémin
Stochastic representation of the Reynolds transport theorem: revisiting large-scale modeling, Comp. and Fluids, 156, pp.456-469, 2017

V. Resseguier, E. Mémin, D. Heitz and B. Chapron
Stochastic modeling and diffusion modes for POD models and small-scale flow analysis, J. of Fluid Mech., 828: 888-917, 2017

V. Resseguier, E. Mémin, B. Chapron

E. Mémin