A fluctuation-dissipation relation for the ocean subject to turbulent atmospheric forcing

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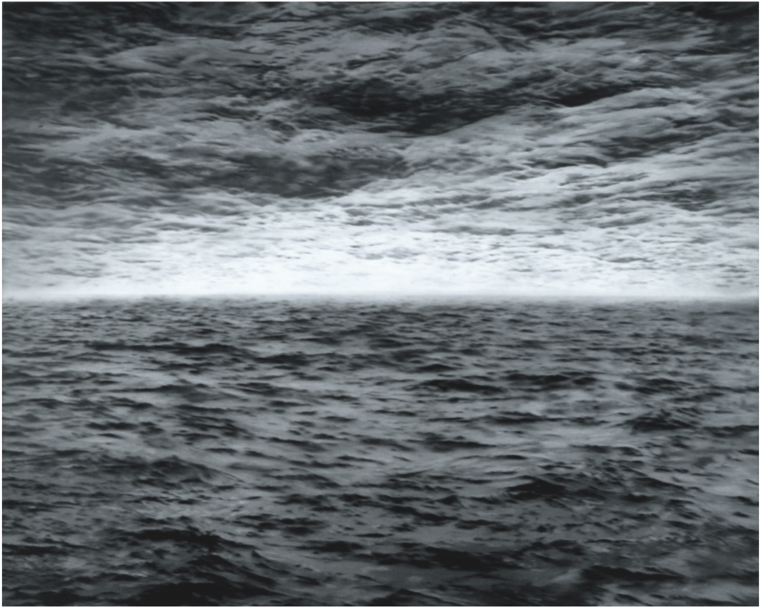
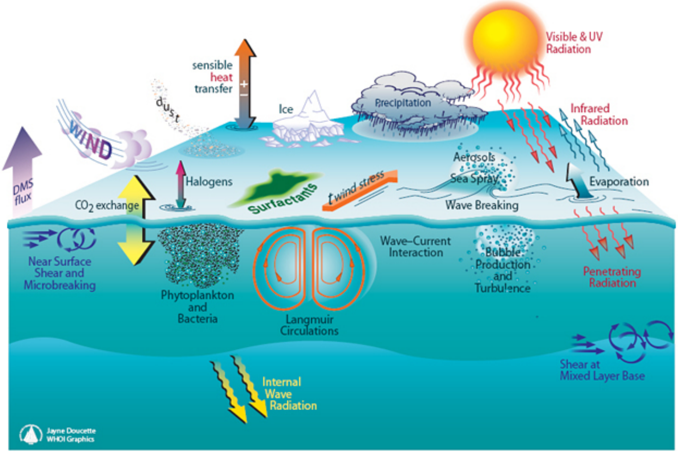
04/04/2019, Potsdam

Air-Sea Interaction

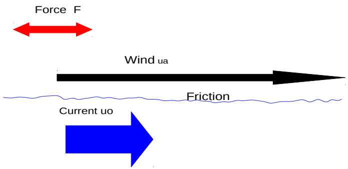


Seestück (G. Richter)

Air-Sea Interaction

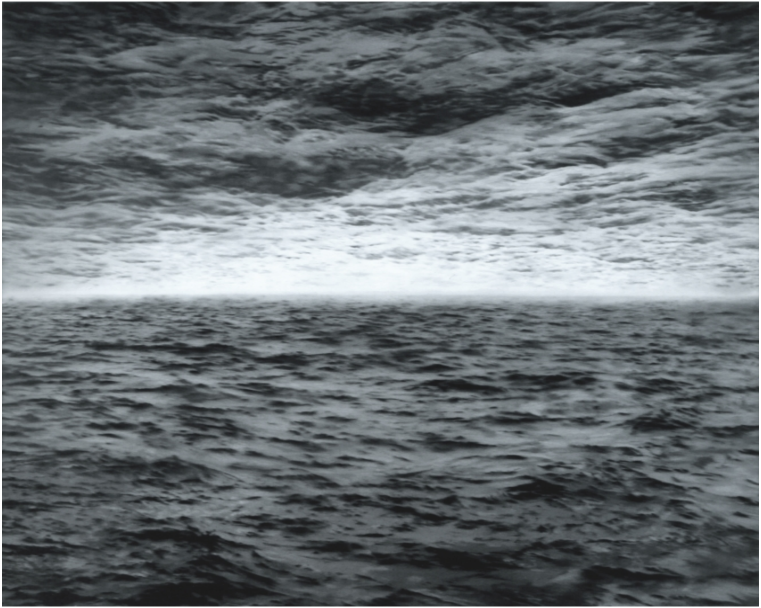


Model



Parameters :

- mass ratio ocean/atmosphere: m
- friction coefficient (nonlinear): c_D



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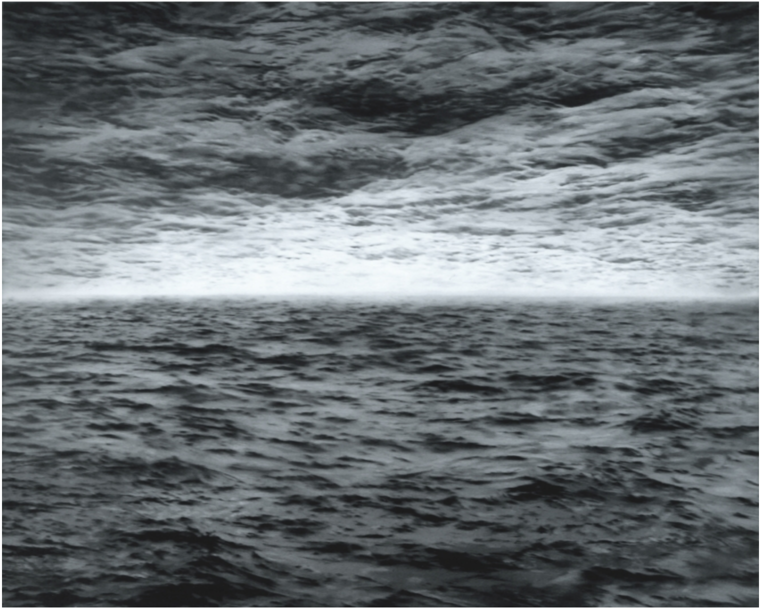
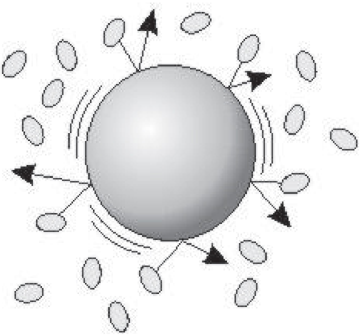
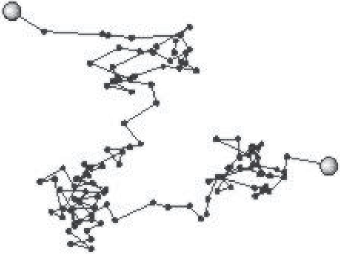


Brownian motion





Einstein relation (1905)

- macroscopic: Stoke's law : $\gamma = \dfrac{6\pi\eta r}{m}$
- microscopic: random walk (1D): D = $\dfrac{\langle x^2 \rangle}{2t} = \dfrac{R}{\gamma^2}$
- equipartition : $\dfrac{k_B T}{m} = \langle u(t)^2 \rangle = \dfrac{R}{\gamma}$

$$D = \frac{k_B T}{\gamma m} = \frac{\mathrm{R} T}{N 6\pi\eta r}$$

Langevin Equation (1908)

$$m\partial_t u(t) = -m\gamma u(t) + F(t)$$

dissipation:	γ	macroscopic	systematic	constant
fluctuationn:	$F(t)$	microscopic	random	$\langle F(t)\rangle = 0$

$$\frac{m}{2}\partial_{tt}x^2 - mu^2 = -\frac{\gamma m}{2}\partial_t x^2 + xF$$

$$\frac{m}{2}\partial_{tt}\langle x^2\rangle - m\langle u^2\rangle = -\frac{m\gamma}{2}\partial_t\langle x^2\rangle + \cancel{\langle xF\rangle}$$

$$\frac{m}{2}\partial_t\langle\partial_t x^2\rangle + \frac{m\gamma}{2}\langle\partial_t x^2\rangle = k_B T$$

$$t \gg \frac{1}{\gamma} \quad \rightarrow \quad \langle\partial_t x^2\rangle = \frac{2k_B T}{m\gamma}$$

Langevin Equation, Itô calculus (1940)

$$u(0) = 0$$

$$u(t) = u(0)e^{-\gamma t} + e^{-\gamma t}\int_0^t F(t')e^{\gamma t'}dt'$$

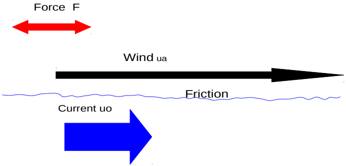
$$\langle u(t)^2\rangle = e^{-2\gamma t}\int_0^t\int_0^t \langle F(t_1)F(t_2)\rangle e^{\gamma(t_1+t_2)}dt_2 dt_1$$

$$\langle F(t_1)F(t_2)\rangle = 2R\delta(t_2 - t_1)$$

Fluctuation dissipation relation:

$$\langle u(t)^2\rangle = \frac{R}{\gamma}$$

Model

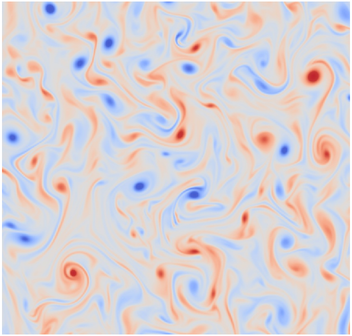


Parameters :

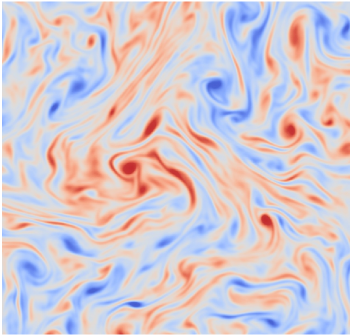
- mass ratio ocean/atmosphere: m
- friction coefficient (nonlinear): c_D

2D Turbulence

Atmos

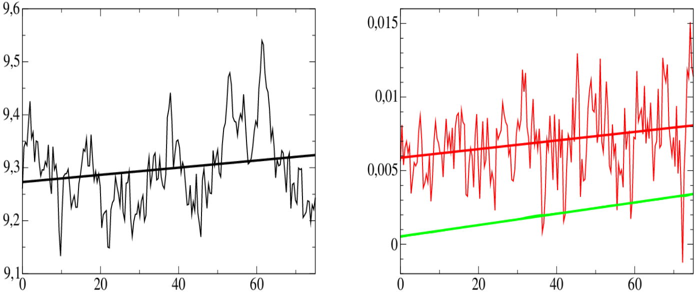


Ocean

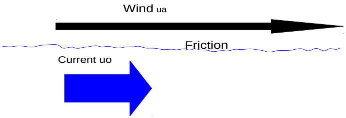


2D Turbulence

$\langle u_a^2 \rangle_A$ $\qquad$ $\langle u_a u_o \rangle_A, \langle u_o^2 \rangle_A$



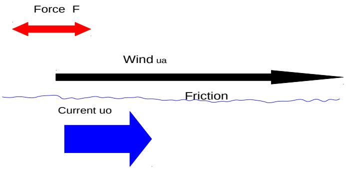
Model



$$\partial_t u_o = -S(u_o - u_a)$$

stat. solution $\leftrightarrow$ 2D turbulence model

Model



$$
\begin{aligned}
\partial_t u_a &= -Sm(u_a - u_o) + F \\
\partial_t u_o &= -S \quad (u_o - u_a)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}\partial_t u_a^2 &= -Sm(u_a^2 - u_a u_o) + F \\
\frac{m}{2}\partial_t u_o^2 &= -Sm(u_o^2 - u_a u_o)
\end{aligned}
$$

$$
\partial_t e = \; = -Sm(u_a - u_o)^2 + F
$$

Linear Local Model

$$\begin{aligned} \partial_t u_s &= -SMu_s + F \\ \partial_t u_t &= F \end{aligned}$$

$$\begin{aligned} u_s(t) &= \int_0^t e^{SM(t'-t)} F(t')dt' \\ u_t(t) &= \int_0^t F(t')dt' \end{aligned}$$

$$\begin{aligned} u_a(t) &= \frac{1}{M}(u_t + mu_s) = \frac{1}{M}\left(\int_0^t F(t')dt' + m\int_0^t e^{SM(t'-t)} F(t')dt'\right) \\ u_o(t) &= \frac{1}{M}(u_t - u_s) = \frac{1}{M}\left(\int_0^t F(t')dt' - \int_0^t e^{SM(t'-t)} F(t')dt'\right) \end{aligned}$$

Linear Local Model : 2nd order moments

$$\begin{aligned}
\langle u_a^2\rangle_\Omega &= \frac{R}{M^2}\left(2t+\frac{4m}{SM}(1-e^{-SMt})+\frac{m^2}{SM}(1-e^{-2SMt})\right)\\
\langle u_o^2\rangle_\Omega &= \frac{R}{M^2}\left(2t-\frac{4}{SM}(1-e^{-SMt})+\frac{1}{SM}(1-e^{-2SMt})\right)\\
\langle u_a u_o\rangle_\Omega &= \frac{R}{M^2}\left(2t+\frac{2(m-1)}{SM}(1-e^{-SMt})-\frac{m}{SM}(1-e^{-2SMt})\right).
\end{aligned}$$

For $t \gg (SM)^{-1}$:

$$\begin{aligned}
\langle (u_a-u_o)^2\rangle_\Omega &= \frac{R}{SM}\\
\langle u_a^2-u_o^2\rangle_\Omega &= \frac{R(M+2)}{SM^2}\\
\langle u_a u_o-u_o^2\rangle_\Omega &= \frac{R}{SM^2}
\end{aligned}$$

Fluctuation Dissipation Relation (FDR)

$$\frac{1}{2}\partial_t \langle u_o^2 \rangle_\Omega = S\langle u_a u_o - u_o^2 \rangle_\Omega = \frac{R(1-e^{-SMt})^2}{M^2}$$

For $t \gg (SM)^{-1}$:

$$\frac{R}{M^2} = \frac{SR}{M^2}\left(2t + \frac{m-2}{SM} - 2t + \frac{3}{SM}\right)$$

Quadratic Local Model

$$\begin{aligned}\partial_t \mathbf{u}_a &= - \quad \tilde{S} m |\mathbf{u}_s| \mathbf{u}_s + \mathbf{F} \\ \partial_t \mathbf{u}_o &= \quad \tilde{S} \; |\mathbf{u}_s| \mathbf{u}_s\end{aligned}$$

with $\mathbf{u}_s = \mathbf{u}_a - \mathbf{u}_o$, $\mathbf{u}_t = \mathbf{u}_a + m\mathbf{u}_o$.

$$\begin{aligned}\partial_t \mathbf{u}_s &= -\tilde{S} M |\mathbf{u}_s| \mathbf{u}_s + \mathbf{F} \\ \partial_t \mathbf{u}_t &= \mathbf{F}\end{aligned}$$

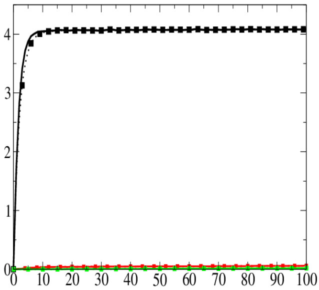
Linear Langevin eq. with eddy friction:

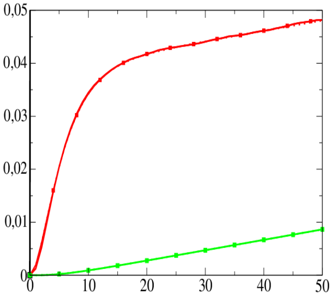
$$\frac{S_{\mathrm{eddy}}}{\tilde{S}} = \frac{\langle (\mathbf{u}_s^2)^{3/2} \rangle}{\langle \mathbf{u}_s^2 \rangle^{3/2}} \langle (\mathbf{u}_s^2)^{1/2} \rangle = \left(\frac{\mu^2 2R}{\tilde{S} M} \right)^{1/3} .$$

$$\mu_{\mathrm{Gaussian}} = \frac{\langle (\mathbf{u}_s^2)^{3/2} \rangle}{\langle \mathbf{u}_s^2 \rangle^{3/2}} = \frac{3\sqrt{\pi}}{4} \approx 1.3293404.$$

Lin. vs. Quadratic Langevin eq.

$$\langle u_a^2 \rangle_A, \langle u_o^2 \rangle_A, \langle u_a u_o \rangle_A$$





$$\mu = \frac{2}{3}\frac{\Gamma(2/3)}{\Gamma(4/3)} \approx 1.2449; (Gaussian) = \frac{3\sqrt{\pi}}{4} \approx 1.329$$

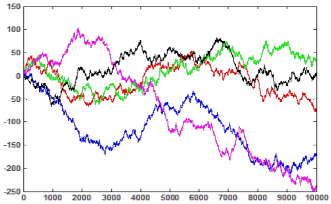
Stochastic differential equation:

Integrating many independent realisation:

$$\partial_t u = F(u, \omega) \quad \text{with}, \quad \omega \in \Omega$$

$\rightarrow$ measure moments :

$$\langle u^n \rangle_\Omega, \qquad \langle f(u) \rangle_\Omega$$



("Lagrangian approach")

Fokker-Planck equation:

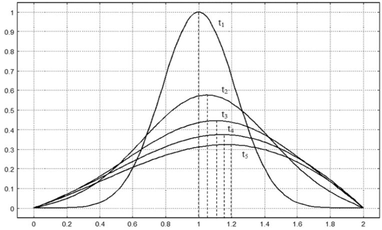
Obtain PDE for the time evolution of the pdf:

$$\partial_t P(u,t) = \partial_u \left(a(u)P(u) + \frac{1}{2}\partial_u \left[b(u)P(u) \right] \right)$$

$\rightarrow$ solve equation if possible and obtain moments by integration:

$$\langle u^n \rangle = \int u^n dP, \qquad \langle f(u) \rangle = \int f(u) dP$$

("Eulerian approach")



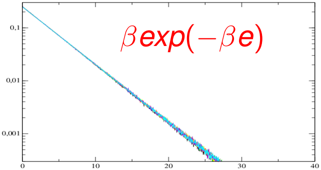
Linear model: SDE $\leftrightarrow$ Fokker-Planck equation:

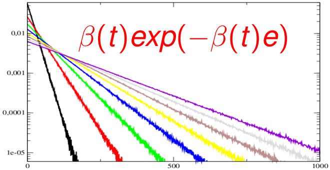
SDE:

$$\begin{aligned} \partial_t \mathbf{u}_s &= -SM\mathbf{u}_s + F \\ \partial_t \mathbf{u}_t &= F \end{aligned}$$

Fokker-Planck

$$\begin{aligned} \partial_t P_s &= \nabla_{uv} \cdot \left[SM\mathbf{u}_s P_s + \frac{1}{2} \nabla_{uv} P_s \right] \\ \partial_t P_t &= \frac{1}{2} \nabla_{uv} \cdot \nabla_{uv} P_t \end{aligned}$$





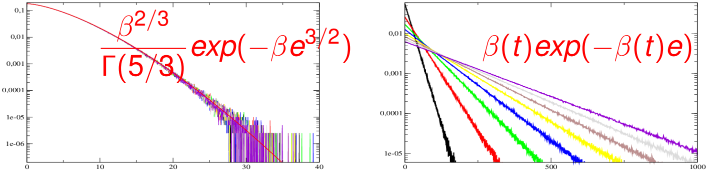
Non-linear model: SDE $\leftrightarrow$ Fokker-Planck equation:

SDE:

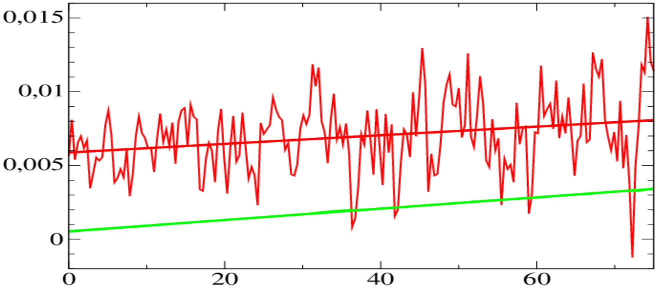
$$
\begin{aligned}
\partial_t \mathbf{u}_s &= - \tilde{S} M |\mathbf{u}_s| \mathbf{u}_s + \mathbf{F} \qquad (1) \\
\partial_t \mathbf{u}_t &= \mathbf{F} \qquad (2)
\end{aligned}
$$

Fokker-Planck

$$
\begin{aligned}
\partial_t P_s &= \nabla_{uv} \cdot \left[\tilde{S} M \mathbf{u}_s u_s P_s + \frac{\nu}{2} \nabla_{uv} P_s \right] \\
\partial_t P_t &= \frac{\nu}{2} \nabla_{uv} \cdot \nabla_{uv} P_s
\end{aligned}
$$



FDR 2D : $\langle \mathbf{u}_o^2 \rangle$, $\langle u_a u_o \rangle$



$$\frac{1}{2}\partial_t \langle u_o^2 \rangle_A = S \langle u_a u_o - u_o^2 \rangle_A$$

$$\tilde{S}_{\text{num}} = \frac{\partial_t \langle \mathbf{u}_o^2 \rangle}{2\mu_{\text{Gauss}} \sqrt{\langle (\mathbf{u}_a - \mathbf{u}_o)^2 \rangle} \langle (\mathbf{u}_a \mathbf{u}_o - \mathbf{u}_o^2) \rangle}$$

$$\frac{\tilde{S}_{\text{num}}}{\tilde{S}} = 0.92$$

Further Results

1) The Fluctuation Dissipation Theorem is established for linear models with white and colored forcing.
$C(t,\Delta t) = \langle \mathbf{x}(t)\mathbf{x}^t(t+\Delta t)\rangle$; $\langle \mathbf{x}(t+\Delta t)\rangle = \chi(t,\Delta t)\bar{\mathbf{x}}$.

$$C(t,\Delta t)C(t,0)^{-1} = \chi(t,\Delta t).$$

2) Fluctuation Theorems are considered.

$$S_{\overline{Z}^\tau}(z) = \ln\left(\frac{f(z)}{f(-z)}\right) = \sigma\tau z,$$

$$\mathrm{Prob}(z_1 < \overline{Z}^\tau < z_2) = \int_{z_1}^{z_2} f(z)dz$$

Conclusions

* The ocean subject to atmospheric forcing obeys a fluctuation dissipation relation.
* Local models (linear and quadratic) can be solved analytically.
* Some of the results from local models can be transposed to fully 2D turbulence models.
* The FDR can be extended to the air-sea system forced by a colored noise.

- Dissipation of non-resolved dynamics is included in models (atmosphere, ocean climate, ...) but not the fluctuations. However, fluctuation-dissipation-relations hold at all levels of the dynamics.
- Consider truely non equilibrium processes (beyond: spin-up, spin-down)