Transform-based Particle Filtering for elliptic Bayesian inverse problems

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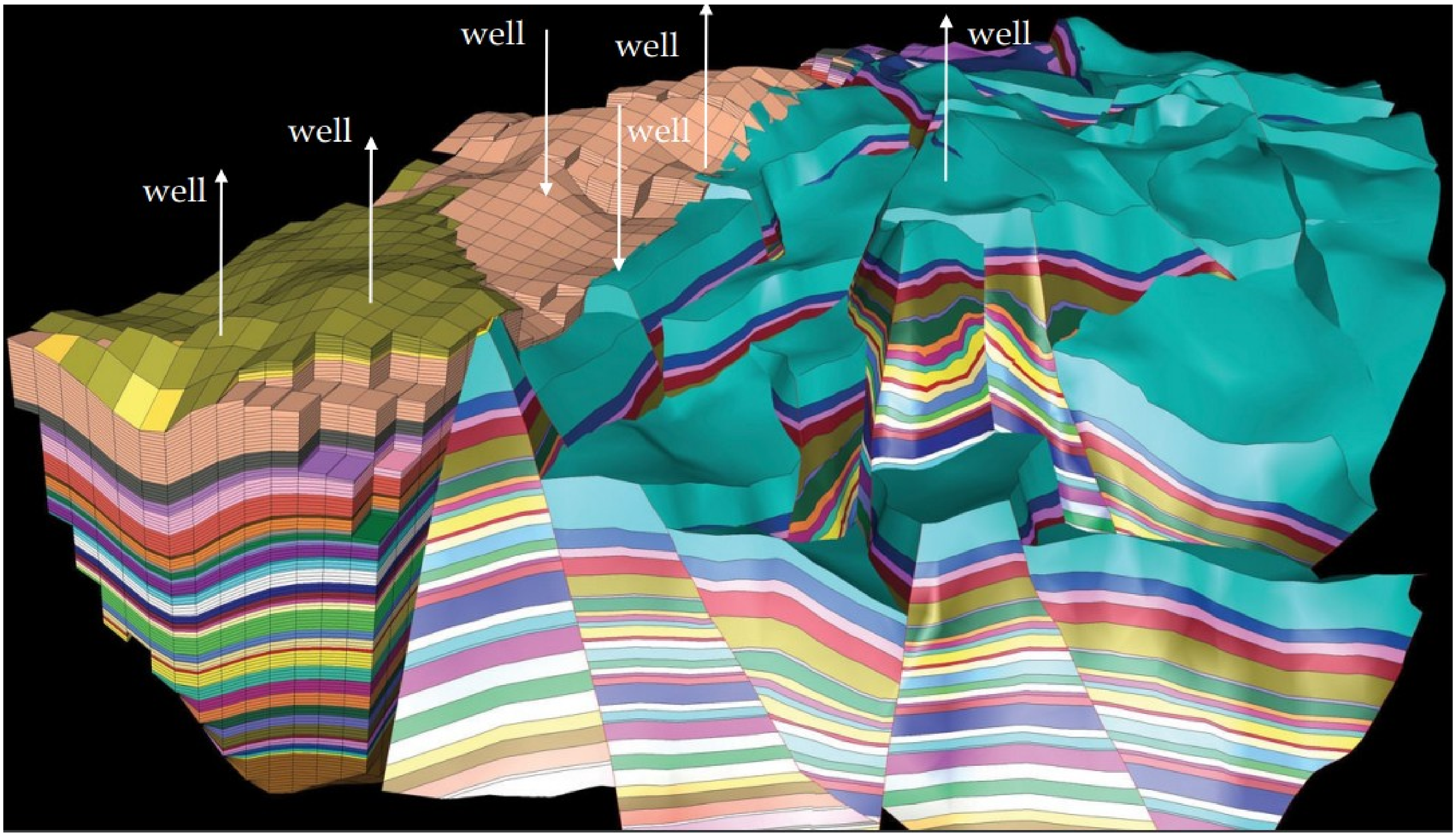
Overview

- Introduction to reservoir modeling
- Data assimilation
  - MCMC
  - Ensemble Kalman filters
  - Ensemble Transform Particle filter
- Test Cases – Description and Results
- Conclusions

Subsurface oil/gas reservoir



Subsurface flow

A simple 2D model for subsurface flow is a diffusion equation

$$-\nabla.(k(x)\nabla P(x))=g(x), \qquad x\in D\subseteq R^2$$

- k, the hydraulic conductivity of the subsurface,
- g, source/sink terms,
- P, the resulting pressure filed of groundwater.

Lack of pressure data leads to uncertainty in the conductivity k.

Data Assimilation

- Instead of a well-posed forward problem of finding pressure from certain permeability, we are faced with an ill-posed inverse problem of finding uncertain random variables from a few pressure measurements. This is an inverse problem of parameter identification.

- Uncertain parameters can be estimated by combining a solution of physical model with measurements by means of data assimilation.

Data Assimilation: MCMC

- The golden standard is Markov Chain Monte Carlo (MCMC).

- MCMC requires very large number of realisations of a model (samples /ensemble members), which is computationally unaffordable for high-dimensional systems.

Brooks, S., Gelman, A., Jones, G. L. and Meng, X.-L., eds. (2011). Handbook of Markov Chain Monte Carlo. CRC Press, Boca Raton, FL.

Data Assimilation: Ensemble Kalman Filter

- Ensemble Kalman Filter (EnKF) became a standard data assimilation method in inverse modeling.

- EnKF assumes Gaussian probabilities, which might not be always the case.

Evensen, G. (2006). Data Assimilation: The Ensemble Kalman Filter. Springer

Data Assimilation: TETPF

- We developed a Tempered Ensemble Transform Particle Filter (TETPF) that does not make such assumptions and applied it to inverse problems.

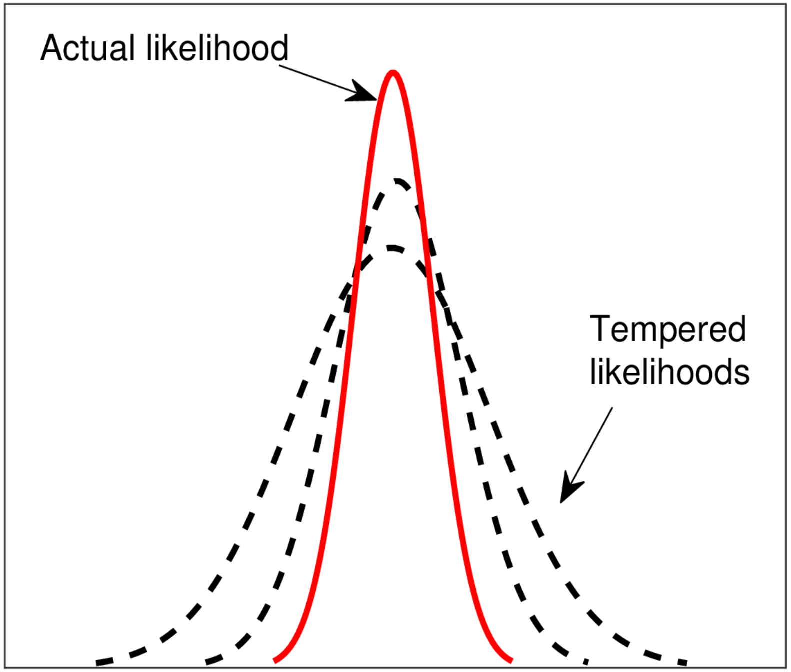
S.Ruchi & S.Dubinkina (2018) and S.Ruchi, S. Dubinkina & M. Iglesias (2018).

- It is based on a data assimilation method of S.Reich.

S. Reich & C. Cotter. (2015). Probabilistic forecasting and data assimilation, Cambridge University Press.

Tempering

- Instead of jumping directly from prior to posterior, a smooth transition among the distribution can lead to stabilization of weights.



$$l(u_{n-1}, y)^{\phi_n - \phi_{n-1}} \propto \exp[-(y - h(u))^2 / 2\alpha\sigma^2]$$

$$\alpha_n = 1/(\phi_n - \phi_{n-1})$$

$$\{\phi_n\}_{n=0}^{q} \qquad 0 < \phi_n < 1$$

$$ESS_n(\phi) = [\sum_{i=1}^{M} (w_{n-1}(\phi))^2]^{-1} = ESS_{thres}$$

Mutation

Select $\beta \in (0,1)$ and an integer N_μ.
for $j = 1, \dots, J$ **do**
Initialize $\nu^{(j)}(0) = \hat{u}_n^{(j)}$
while $\alpha \leq N_\mu$ **do**
(1) **pcN proposal**. Propose u_{prop} from

$$u_{\text{prop}} = \sqrt{1-\beta^2}\nu^{(j)}(\alpha) + (1-\sqrt{1-\beta^2})m + \beta\xi, \qquad \text{with } \xi \sim N(0,C)$$

(2) Set $\nu^{(j)}(\alpha+1) = u_{\text{prop}}$ with probability $a(\nu^{(j)}(\alpha), u)$ and $\nu^{(j)}(\alpha+1) = \nu^{(j)}(\alpha)$ with probability $1 - a(\nu^{(j)}(\alpha), u)$, where

$$a(u,v) = \min\left\{1, \frac{l(u,y)^{\phi_n}}{l(v,y)^{\phi_n}}\right\}$$

(3) $\alpha \leftarrow \alpha + 1$
end while
end for

TETPF-pCN

Let $\{u_0^{(j)}\}_{j=1}^J \sim \mu_0$ be the initial ensemble of J particles.
Define the tunable parameters J_{thresh} and N_μ.
Set $n = 0$ and $\phi_0 = 0$
while $\phi_n < 1$ **do**
    $n \to n + 1$
    Compute the likelihood $l(u_{n-1}^{(j)}, y)$ (for $j = 1, \ldots, J$)
    Compute the tempering parameter ϕ_n:
    if $\min_{\phi \in (\phi_{n-1}, 1)} \text{ESS}_n(\phi) > J_{\text{thresh}}$ **then**
        set $\phi_n = 1$.
    else
        compute ϕ_n such that $\text{ESS}_n(\phi) \approx J_{\text{thresh}}$
        using a bisection algorithm on $(\phi_{n-1}, 1]$.
    end if
    Computing weights from expression $W_n^{(j)} \equiv \mathcal{W}_{n-1}^{(j)}[\phi_n]$
    Resample based on optimal transport. Compute $d_{ij} = ||u_{n-1}^{(i)} - u_{n-1}^{(j)}||^2$ (for $i, j = 1, \ldots, J$). Supply $\{d_{ij}\}_{i,j=1}^J$ and $\{W_n^{(j)}\}_{j=1}^J$ to the Earth's moving distances algorithm of Pele & Werman. The output is the coupling $\{T_{ij}^*\}_{i,j=1}^J$.
    Compute new samples $\hat{u}_n^{(j)}$ and set $W_n^{(j)} = \frac{1}{J}$.
    Mutation. Sample $u_n^{(j)} \sim \mathcal{K}_n(\hat{u}_n^{(j)}, \cdot)$
end while
Approximate μ_n by $\mu_n^J \equiv \frac{1}{J} \sum_{j=1}^J \delta_{u_{n,r}^{(j)}}$

Test Case I: Gaussian probability

Assume a steady-state single-phase 2D model for subsurface flow

$$-\nabla .(k(x)\nabla P(x)) = g(x), \qquad x = (x_1, x_2) \in D$$

- the physical domain, $D = [0,6] \times [0,6]$

- $$g(x_1, x_2) = \begin{cases} 0 & if\ 0 < x_2 < 4 \\ 137 & if\ 4 < x_2 < 5 \\ 274 & if\ 5 < x_2 < 6 \end{cases}$$

- boundary conditions, $P(x_1, 0) = 100; \frac{\partial P}{dx}(6, x_2) = 0;$

$$-k\frac{\partial P}{dx}(0, x_2) = 500; \frac{\partial P}{dy}(x_1, 6) = 0$$

Test Case I: Gaussian probability

- We consider smoothed point observation defined by

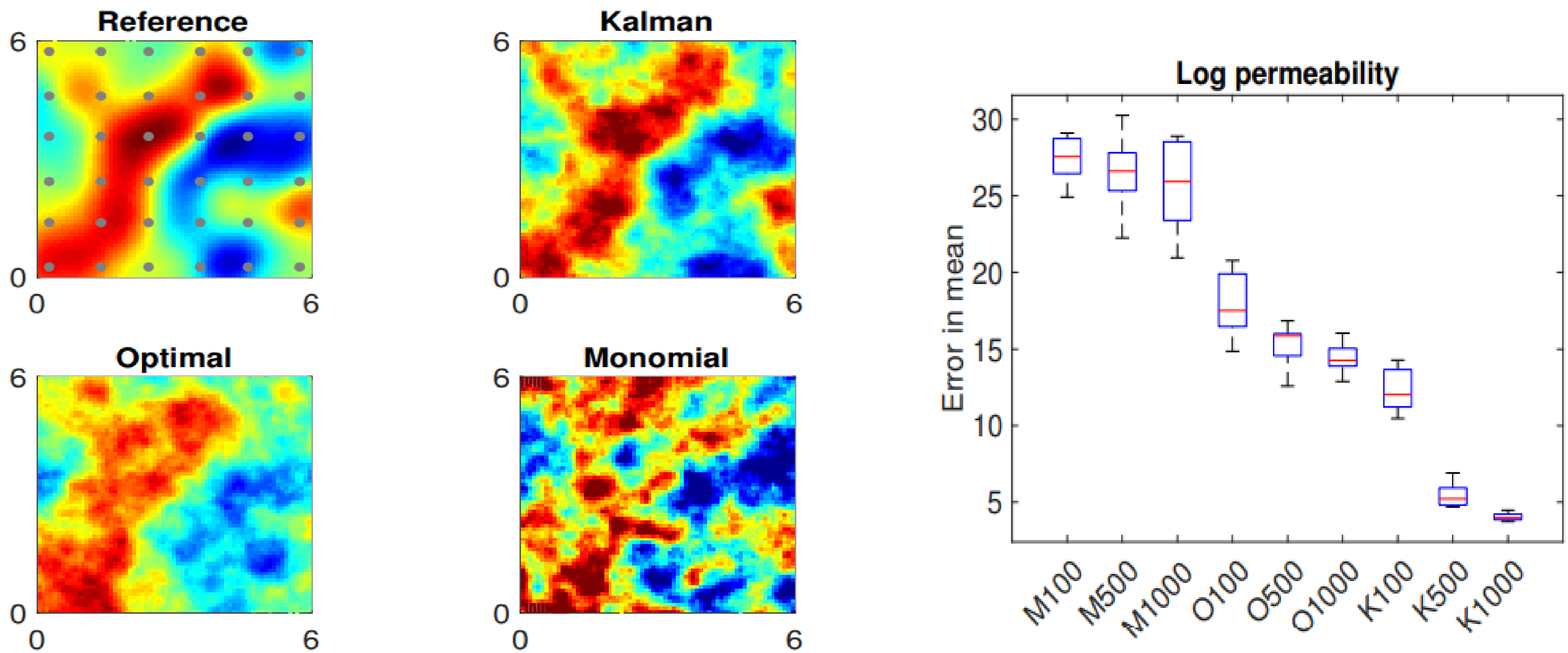
$$\ell_i(p)=\frac{1}{2\pi\varepsilon^2}\int_D \exp\left(\frac{-|x-x_j^2|}{2\varepsilon^2}\right)P(x)\,dx$$

and define a forward map by $G(k)=(\ell_1(p),\dots,\ell_M(p))$

- For this model we simply consider the parameter as natural logarithm of k, i.e. $u(x)=\log k(x)$

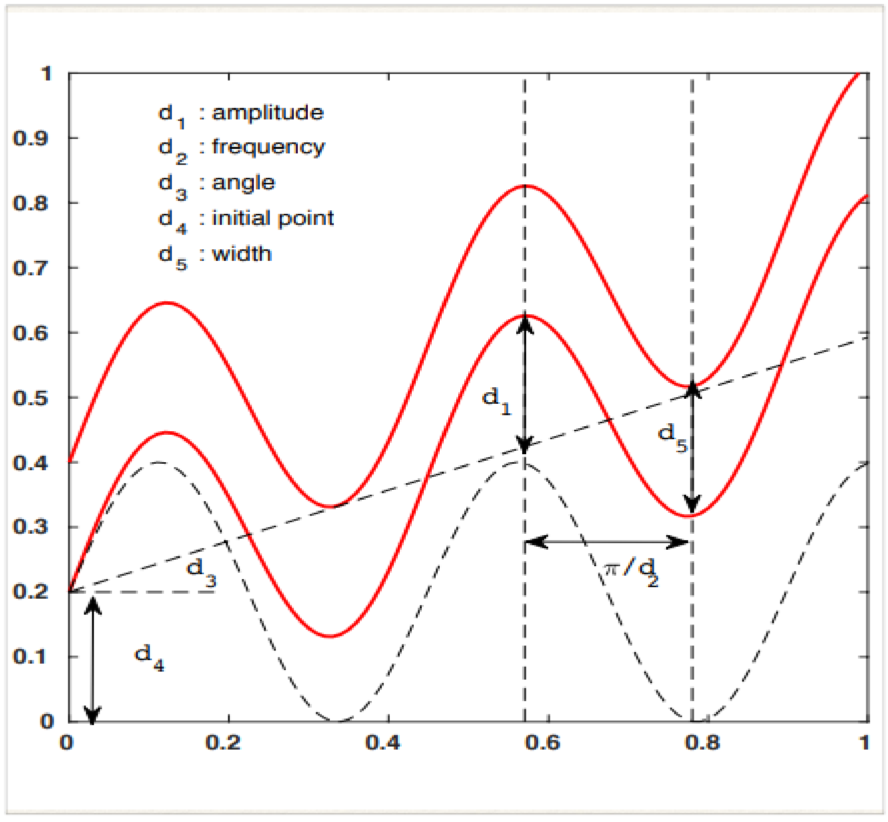
- We consider Gaussian distributed log permeability.

Test Case I: Gaussian probability



- EnKF gives more accurate estimations of the mean field than TETPF.

Test Case II: Bimodal probability



- We consider a channelized domain: a channel with different permeability is situated in the domain.

Test Case II: Bimodal probability

- This model consists of parameterization of permeability of the form

$$k(x)=\exp(u_1(x))\chi_{D_i}(x)+\exp(u_2(x))\chi_{D_o}(x)$$

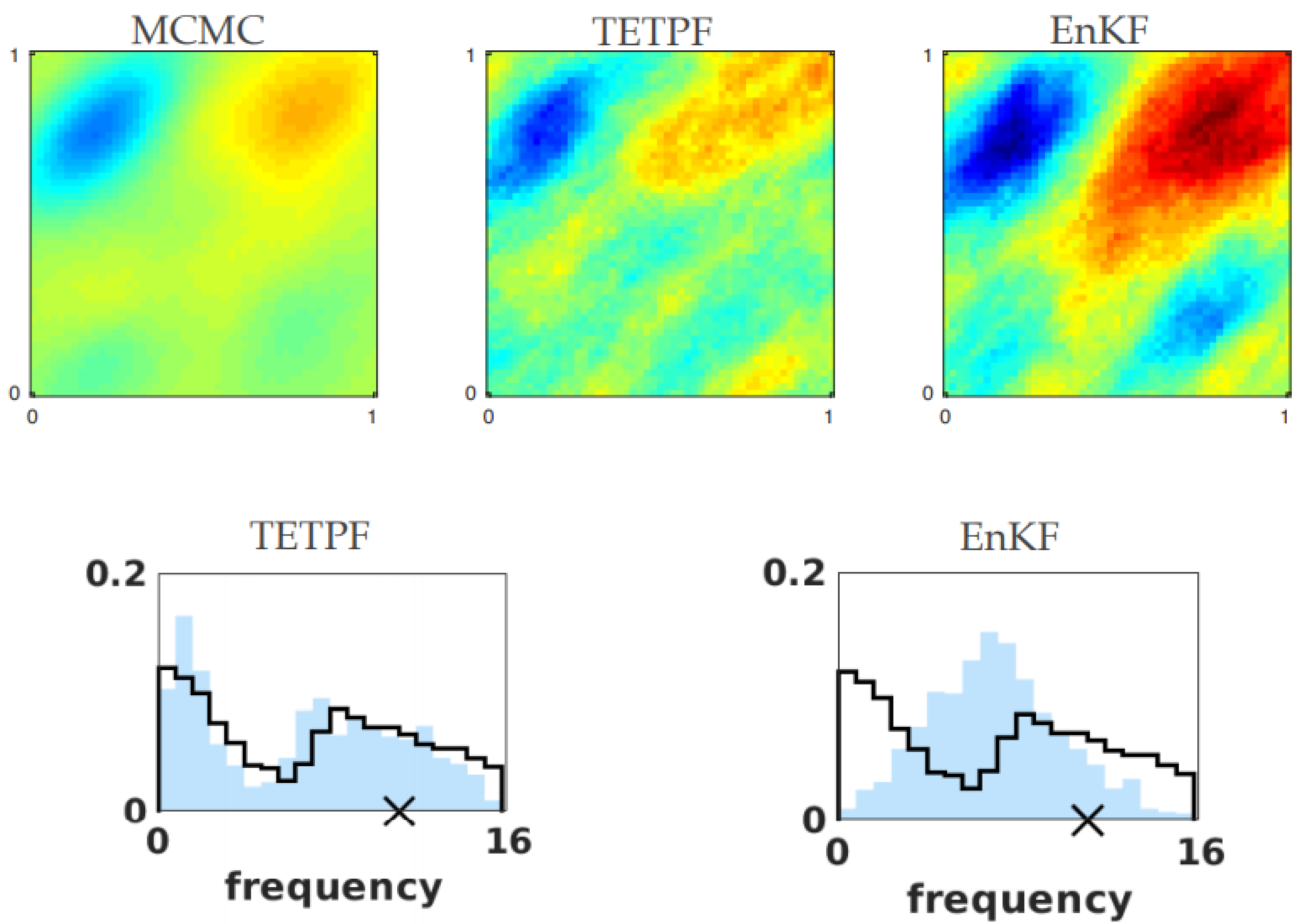
where $k_1=\exp(u_1(x));\quad k_2=\exp(u_2(x))$

- The geometry of the channel is parameterized by five parameters $\{d_i\}_{i=1}^{5}$; amplitude, frequency, angle, initial point, width.

- The lower boundary of channel is; $x_2=d_1\sin(d_2 x_1)+\tan(d_3)x_1+d_4$ and the upper boundary is $x_2+d_5.$

- the parameters of interest are comprised in $u=(d_1,d_2,d_3,d_4,d_5,u_1,u_2)$

Test Case II: Bimodal probability



Conclusions

Accurate estimations can be obtained by

- EnKF, when everything is Gaussian;
- MCMC, when everything is low-dimensional.
- TETPF, when everything is high-dimensional and non-Gaussian.

Questions?