Stability and uniform fluctuation estimates of Ensemble Kalman-Bucy filters

P. Del Moral

Data Assimilation Research Center, Potsdam Univ. 2018

Synthesis joint works:

- `AAP-17 (Unif. EnKBF)+ Tugaut.`
- `SIAM Control & Opt.-17 (Unif. En-EKBF)+ Kurtzmann, Tugaut.`
- `SIAM Control & Opt.-17/18 (Stability KBF)+ Bishop`
- `SPA-17 (Perturbation KB)+ Bishop, S. Pathiraja.`
- `EJP-18 (Stability EKBF)+ Kurtzmann, Tugaut.`
- `AAP-18/19 (1d-case)+ Bishop, Kamatani, Rémillard.`
- `Arxiv (Perturbation Stoch. Riccati)+ Bishop, A. Niclas.`
- `Arxiv (Stability Stoch. Riccati)+ Bishop.`

Kalman-Bucy filter

McKean-Vlasov interpretations

Stability of Kalman-Bucy diffusions

Uniform fluctuation estimates

Nonlinear models

Kalman-Bucy filter

Linear+Gaussian filtering problem

$$\left\{\begin{array}{lcl} dX_t &=& A\,X_t\,dt \;+\; R^{1/2}\,dW_t \;\in \mathbb{R}^r \\ dY_t &=& C\,X_t\,dt \;+\; \Sigma^{1/2}\,dV_t \qquad \rightsquigarrow \mathcal{F}_t := \sigma(Y_s,\; s \leq t). \end{array}\right.$$

Kalman-Bucy filter

Linear+Gaussian filtering problem

$$\left\{\begin{array}{lcl} dX_t &=& A\,X_t\,dt\;+\;R^{1/2}\,dW_t\;\in\mathbb{R}^r \\ dY_t &=& C\,X_t\,dt\;+\;\Sigma^{1/2}\,dV_t \qquad \rightsquigarrow \mathcal{F}_t := \sigma(Y_s,\;s\leq t). \end{array}\right.$$

Optimal $\mathbb{L}_2$-filter = Kalman-Bucy filter

$$\widehat{X}_t := \mathbb{E}(X_t \mid \mathcal{F}_t) \quad \text{and} \quad P_t := \mathbb{E}\left((X_t - \mathbb{E}(X_t \mid \mathcal{F}_t))\,(X_t - \mathbb{E}(X_t \mid \mathcal{F}_t))'\right)$$

Kalman-Bucy filter

Linear+Gaussian filtering problem

$$\left\{\begin{array}{lcl} dX_t & = & A\, X_t\, dt \ + \ R^{1/2}\, dW_t \ \in \mathbb{R}^r \\ dY_t & = & C\, X_t\, dt \ + \ \Sigma^{1/2}\, dV_t \end{array}\right. \quad \rightsquigarrow \mathcal{F}_t := \sigma(Y_s,\ s \leq t).$$

Optimal $\mathbb{L}_2$-filter = Kalman-Bucy filter

$$\widehat{X}_t := \mathbb{E}(X_t \mid \mathcal{F}_t) \quad \text{and} \quad P_t := \mathbb{E}\left((X_t - \mathbb{E}(X_t \mid \mathcal{F}_t))\,(X_t - \mathbb{E}(X_t \mid \mathcal{F}_t))'\right)$$

$$\Downarrow$$

State estimate

$$d\widehat{X}_t = A\ \widehat{X}_t\ dt + P_t\ C'\Sigma^{-1}\ \left(dY_t - C\widehat{X}_t\ dt\right)$$

Kalman-Bucy filter

Linear+Gaussian filtering problem

$$\left\{\begin{array}{lcl} dX_t & = & A\ X_t\ dt\ +\ R^{1/2}\ dW_t\ \in \mathbb{R}^r \\ dY_t & = & C\ X_t\ dt\ +\ \Sigma^{1/2}\ dV_t \qquad \rightsquigarrow \mathcal{F}_t := \sigma(Y_s,\ s \leq t). \end{array}\right.$$

Optimal $\mathbb{L}_2$-filter = Kalman-Bucy filter

$$\widehat{X}_t := \mathbb{E}(X_t \mid \mathcal{F}_t) \quad \text{and} \quad P_t := \mathbb{E}\left((X_t - \mathbb{E}(X_t \mid \mathcal{F}_t))\,(X_t - \mathbb{E}(X_t \mid \mathcal{F}_t))'\right)$$

$$\Downarrow$$

State estimate

$$d\widehat{X}_t = A\ \widehat{X}_t\ dt + P_t\ C'\Sigma^{-1}\ \left(dY_t - C\widehat{X}_t\ dt\right)$$

with the gain given by the matrix Riccati equation

$$\partial_t P_t = \mathrm{Ricc}(P_t) := AP_t + P_tA' - P_t\mathbf{S}P_t + R \quad \text{with} \quad \mathbf{S} := \mathbf{C}'\mathbf{\Sigma}\mathbf{C}$$

Kalman-Bucy filter

McKean-Vlasov interpretations
    3 classes of algorithms
    Mean field/Ensemble Kalman-Bucy filter
    Dyson equation
    The 1d case

Stability of Kalman-Bucy diffusions

Uniform fluctuation estimates

Nonlinear models

Reformulation $\rightsquigarrow$ Nonlinear Kalman-Bucy diffusion

$\Longleftrightarrow$ **McKean-Vlasov type diffusions $\overline{X}_t$ such that**

$$\eta_t := \mathrm{Law}(\overline{X}_t \mid \mathcal{F}_t) \; = \; \mathcal{N}\left[\widehat{X}_t, P_t\right]$$

$\rightsquigarrow$ **Interacting with their conditional mean and covariance matrices**

$$\mathcal{P}_{\eta_t} = \eta_t\left[(e - \eta_t(e))(e - \eta_t(e))'\right] \quad \text{with} \quad e(x) := x.$$

3 classes of McKean-Vlasov type diffusions

1) "Vanilla EnKF" ($\rightsquigarrow$ discrete time - Evensen 94)

$$d\overline{X}_t = A\,\overline{X}_t dt + R^{1/2}\, d\overline{W}_t + \mathcal{P}_{\eta_t} C'\Sigma^{-1}\, \left[dY_t - \left(C\,\overline{X}_t dt + \Sigma^{1/2}\, d\overline{V}_t\right)\right]$$

3 classes of McKean-Vlasov type diffusions

1) "Vanilla EnKF" ($\rightsquigarrow$ discrete time - Evensen 94)

$$d\overline{X}_t = A\,\overline{X}_t dt + R^{1/2}\,d\overline{W}_t + \mathcal{P}_{\eta_t} C'\Sigma^{-1}\left[dY_t - \left(C\,\overline{X}_t dt + \Sigma^{1/2}\,d\overline{V}_t\right)\right]$$

2) "deterministic EnKF" ($\rightsquigarrow$ discrete time - Sakov-Oke 08)

$$d\overline{X}_t = A\,\overline{X}_t dt + R^{1/2}\,d\overline{W}_t + \mathcal{P}_{\eta_t} C'\Sigma^{-1}\left[dY_t - C\left(\frac{\overline{X}_t + \eta_t(e)}{2}\right)dt\right]$$

3 classes of McKean-Vlasov type diffusions

1) "Vanilla EnKF" ($\rightsquigarrow$ discrete time - Evensen 94)

$$d\overline{X}_t = A\,\overline{X}_t dt + R^{1/2}\,d\overline{W}_t + \mathcal{P}_{\eta_t} C'\Sigma^{-1}\left[dY_t - \left(C\,\overline{X}_t dt + \Sigma^{1/2}\,d\overline{V}_t\right)\right]$$

2) "deterministic EnKF" ($\rightsquigarrow$ discrete time - Sakov-Oke 08)

$$d\overline{X}_t = A\,\overline{X}_t dt + R^{1/2}\,d\overline{W}_t + \mathcal{P}_{\eta_t} C'\Sigma^{-1}\left[dY_t - C\left(\frac{\overline{X}_t + \eta_t(e)}{2}\right) dt\right]$$

3) Pure transport equation (Reich-Cotter 13):

$$d\overline{X}_t = A\,\overline{X}_t dt$$

$$+\frac{1}{2}\left(R - \mathcal{P}_{\eta_t} S \mathcal{P}_{\eta_t}\right)\mathcal{P}_{\eta_t}^{-1}(\overline{X}_t - \eta_t(e))\,dt + \mathcal{P}_{\eta_t} C'\Sigma^{-1}\,[dY_t - C\eta_t(e)dt]$$

3 classes of McKean-Vlasov type diffusions

1) "Vanilla EnKF" ($\rightsquigarrow$ discrete time - Evensen 94)

$$d\overline{X}_t = A\,\overline{X}_t dt + R^{1/2}\,d\overline{W}_t + \mathcal{P}_{\eta_t} C'\Sigma^{-1}\,\left[dY_t - \left(C\,\overline{X}_t dt + \Sigma^{1/2}\,d\overline{V}_t\right)\right]$$

2) "deterministic EnKF" ($\rightsquigarrow$ discrete time - Sakov-Oke 08)

$$d\overline{X}_t = A\,\overline{X}_t dt\; +\; R^{1/2}\,d\overline{W}_t + \mathcal{P}_{\eta_t} C'\Sigma^{-1}\,\left[dY_t - C\,\left(\frac{\overline{X}_t + \eta_t(e)}{2}\right)\,dt\right]$$

3) Pure transport equation (Reich-Cotter 13):

$$d\overline{X}_t = A\,\overline{X}_t dt$$

$$+\,\frac{1}{2}\,(R - \mathcal{P}_{\eta_t} S \mathcal{P}_{\eta_t})\mathcal{P}_{\eta_t}^{-1}(\overline{X}_t - \eta_t(e))\,dt + \mathcal{P}_{\eta_t} C'\Sigma^{-1}\,[dY_t - C\eta_t(e)dt]$$

$\oplus$ **Many others, adding** $\mathcal{Q}_{\eta_t}\mathcal{P}_{\eta_t}^{-1}(\overline{X}_t - \eta_t(e))\,dt$ **for any** $\mathcal{Q}'_{\eta_t} = -\mathcal{Q}_{\eta_t}$.

The Ensemble Kalman-Bucy filter

(Case 1) Mean field interpretation $\rightsquigarrow$ $N+1$ interacting diffusions

$$d\xi_t^i = A\,\xi_t^i dt + R^{1/2} d\overline{W}_t^i + p_t C' \Sigma^{-1}\left[dY_t - \left(C\xi_t^i dt + \Sigma^{1/2}\, d\overline{V}_t^i\right)\right]$$

with the rescaled particle covariance matrices

$$p_t := \left(1+\frac{1}{N}\right)\ \mathcal{P}_{\eta_t^N} = \frac{1}{N}\sum_{1\leq i\leq N+1} \left(\xi_t^i - m_t\right)\left(\xi_t^i - m_t\right)'$$

and the empirical measures

$$\eta_t^N := \frac{1}{N+1}\sum_{1\leq i\leq N+1} \delta_{\xi_t^i} \quad \text{and the sample mean} \quad m_t := \frac{1}{N+1}\sum_{1\leq i\leq N+1} \xi_t^i.$$

The Ensemble Kalman-Bucy filter

(Case 1) Mean field interpretation $\rightsquigarrow$ $N+1$ interacting diffusions

$$d\xi_t^i = A\,\xi_t^i dt + R^{1/2} d\overline{W}_t^i + p_t C' \Sigma^{-1} \left[dY_t - \left(C\xi_t^i dt + \Sigma^{1/2}\, d\overline{V}_t^i \right) \right]$$

with the rescaled particle covariance matrices

$$p_t := \left(1 + \frac{1}{N}\right) \mathcal{P}_{\eta_t^N} = \frac{1}{N} \sum_{1 \le i \le N+1} \left(\xi_t^i - m_t\right) \left(\xi_t^i - m_t\right)'$$

and the empirical measures

$$\eta_t^N := \frac{1}{N+1} \sum_{1 \le i \le N+1} \delta_{\xi_t^i} \quad \text{and the sample mean} \quad m_t := \frac{1}{N+1} \sum_{1 \le i \le N+1} \xi_t^i.$$

where is the Riccati equation?

Th1: The EnKF equations

The EnKF equation

$$dm_t = A\ m_t dt + p_t\ C'\Sigma^{-1}\ (dY_t - Cm_t\ dt) + \frac{1}{\sqrt{N+1}}\ d\overline{M}_t$$

with an **r**-dimensional martingale $\overline{M}_t = \left(\overline{M}_t(k)\right)_{1\leq k\leq r}$ with angle-brackets

$$\partial_t\langle\overline{M}\mid\ \otimes\ \mid\overline{M}\rangle_t = U + p_t V p_t.$$

With

1) $(U,V) = (R,S)$ 2) $(U,V) = (R,0)$ 3) $(U,V) = (0,0)$

The particle/ensemble Riccati equation

$$dp_t = \mathrm{Ricc}(p_t)\ dt + \frac{1}{\sqrt{N}}\ dM_t$$

Symmetric matrix-valued martingale $M_t = (M_t(k,l))_{1\leq k,l\leq r}$

The particle/ensemble Riccati equation

$$dp_t = \mathrm{Ricc}(p_t)\ dt + \frac{1}{\sqrt{N}}\ dM_t$$

Symmetric matrix-valued martingale $M_t = (M_t(k,l))_{1\leq k,l\leq r}$

Angle brackets $=$ **the Wick-type formula** $((\centerdot \otimes \ \centerdot)^{\sharp}:=$ **entrywise)**

$$\partial_t \left\langle M \mid \otimes \mid M \right\rangle_t^{\sharp} = p_t \otimes_{sym} (U + p_t V p_t)$$

The particle/ensemble Riccati equation

$$dp_t = \mathrm{Ricc}(p_t)\ dt + \frac{1}{\sqrt{N}}\ dM_t$$

Symmetric matrix-valued martingale $M_t = (M_t(k,l))_{1\leq k,l\leq r}$

Angle brackets $=$ **the Wick-type formula** $((\cdot \otimes \ \cdot)^\sharp :=$ **entrywise)**

$$\partial_t \left\langle M \,\middle|\, \otimes \,\middle|\, M\right\rangle_t^\sharp = p_t \otimes_{sym} (U + p_t V p_t)$$

$\rightsquigarrow V > 0 \Longrightarrow CUBIC \Rightarrow$ **Explosive Euler-discrete scheme**

The particle/ensemble Riccati equation

$$dp_t = \mathrm{Ricc}(p_t)\ dt + \frac{1}{\sqrt{N}}\ dM_t$$

Symmetric matrix-valued martingale $M_t = (M_t(k,l))_{1\leq k,l\leq r}$

Angle brackets $=$ the Wick-type formula $((\centerdot \otimes \ \centerdot)^{\sharp}$:= entrywise)

$$\partial_t \langle M \mid \otimes \mid M\rangle_t^{\sharp} = p_t \otimes_{sym} (U + p_t V p_t)$$

$\rightsquigarrow V > 0 \Longrightarrow CUBIC \Rightarrow$ **Explosive Euler-discrete scheme**

Orthogonality property

$$\forall 1 \leq k, l, l' \leq r \qquad \left\langle M(k,l), \overline{M}(l')\right\rangle_t = 0.$$

In terms of random matrices with $\epsilon := \frac{2}{\sqrt{N}}$

$$\mathcal{W}_t = (\mathcal{W}_t(i,j))_{1\leq i,j\leq r} \quad \text{independent Brownian motions}$$

$$\downarrow$$

$$dp_t \overset{law}{=} [Ap_t + p_t A' + R - p_t S p_t] \ dt + \epsilon \ \left(p_t^{1/2} d\mathcal{W}_t \left(U + p_t V p_t\right)^{1/2}\right)_{sym}$$

- Case 3 $\Longleftrightarrow (U, V) = (0, 0)$

In terms of random matrices with $\epsilon := \frac{2}{\sqrt{N}}$

$$\mathcal{W}_t = (\mathcal{W}_t(i,j))_{1\leq i,j\leq r} \quad \text{independent Brownian motions}$$

$$\downarrow$$

$$dp_t \overset{law}{=} [Ap_t + p_t A' + R - p_t S p_t] \; dt + \epsilon \; \left(p_t^{1/2} d\mathcal{W}_t \, (U + p_t V p_t)^{1/2}\right)_{sym}$$

- Case 3 $\Longleftrightarrow (U, V) = (0,0) \rightsquigarrow$ Deterministic Riccati equation

In terms of random matrices with $\epsilon := \frac{2}{\sqrt{N}}$

$$\mathcal{W}_t = (\mathcal{W}_t(i,j))_{1\leq i,j\leq r} \quad \text{independent Brownian motions}$$

$$\downarrow$$

$$dp_t \overset{law}{=} [Ap_t + p_t A' + R - p_t S p_t] \ dt + \epsilon \ \left(p_t^{1/2} d\mathcal{W}_t \left(U + p_t V p_t\right)^{1/2}\right)_{sym}$$

- Case 3 $\iff (U,V) = (0,0) \rightsquigarrow$ Deterministic Riccati equation
- $S = 0 = V$

In terms of random matrices with $\epsilon := \frac{2}{\sqrt{N}}$

$$\mathcal{W}_t = (\mathcal{W}_t(i,j))_{1\leq i,j\leq r} \quad \text{independent Brownian motions}$$

$$\downarrow$$

$$dp_t \overset{law}{=} [Ap_t + p_t A' + R - p_t S p_t] \ dt + \epsilon \ \left(p_t^{1/2} d\mathcal{W}_t \left(U + p_t V p_t\right)^{1/2} \right)_{sym}$$

- Case 3 $\Longleftrightarrow (U,V) = (0,0) \rightsquigarrow$ Deterministic Riccati equation
- $S = 0 = V \rightsquigarrow$ Wishart process

In terms of random matrices with $\epsilon := \frac{2}{\sqrt{N}}$

$$\mathcal{W}_t = (\mathcal{W}_t(i,j))_{1\leq i,j\leq r} \quad \text{independent Brownian motions}$$

$$\downarrow$$

$$dp_t \overset{law}{=} [Ap_t + p_t A' + R - p_t S p_t]\ dt + \epsilon\ \left(p_t^{1/2} d\mathcal{W}_t\, (U + p_t V p_t)^{1/2}\right)_{sym}$$

- Case 3 $\Longleftrightarrow (U,V) = (0,0) \rightsquigarrow$ Deterministic Riccati equation
- $S = 0 = V \rightsquigarrow$ Wishart process
- $(A,R,S) = (\alpha\ I, \beta\ I, \gamma\ I)$ and $(U,V) = (I,0)$

In terms of random matrices with $\epsilon := \frac{2}{\sqrt{N}}$

$$\mathcal{W}_t = (\mathcal{W}_t(i,j))_{1\leq i,j\leq r} \quad \text{independent Brownian motions}$$

$$\downarrow$$

$$dp_t \overset{law}{=} [Ap_t + p_t A' + R - p_t S p_t]\ dt + \epsilon\ \left(p_t^{1/2} d\mathcal{W}_t \left(U + p_t V p_t\right)^{1/2}\right)_{sym}$$

- Case 3 $\Longleftrightarrow (U,V) = (0,0) \rightsquigarrow$ Deterministic Riccati equation
- $S = 0 = V \rightsquigarrow$ Wishart process
- $(A,R,S) = (\alpha\ I, \beta\ I, \gamma\ I)$ and $(U,V) = (I,0) \rightsquigarrow$ Dyson equation

In terms of random matrices with $\epsilon := \frac{2}{\sqrt{N}}$

$$\mathcal{W}_t = (\mathcal{W}_t(i,j))_{1\leq i,j\leq r} \quad \text{independent Brownian motions}$$

$$\downarrow$$

$$dp_t \overset{law}{=} [Ap_t + p_t A' + R - p_t S p_t] \; dt + \epsilon \; \left(p_t^{1/2} d\mathcal{W}_t \left(U + p_t V p_t\right)^{1/2} \right)_{sym}$$

- Case 3 $\Longleftrightarrow (U, V) = (0,0) \rightsquigarrow$ Deterministic Riccati equation
- $S = 0 = V \rightsquigarrow$ Wishart process
- $(A, R, S) = (\alpha \; I, \beta \; I, \gamma \; I)$ and $(U, V) = (I, 0) \rightsquigarrow$ Dyson equation

$p_t \rightsquigarrow$ *non colliding eigenvalues* $\lambda_r(t) < \ldots < \lambda_2(t) < \lambda_1(t)$

$$d\lambda_i(t) = \left[2\alpha\lambda_i(t) + \beta - \lambda_i(t)^2\gamma + \frac{\epsilon^2}{4} \sum_{j\neq i} \frac{\lambda_i(t) + \lambda_j(t)}{\lambda_i(t) - \lambda_j(t)} \right] dt + \epsilon \; \sqrt{\lambda_i(t)} \; dW_t^i$$

The 1d case $\rightsquigarrow$ Closed form Riccati semigroups

Deterministic Riccati P_t on $\mathbb{R}_+$: $\mathrm{Ricc}(\varpi_\pm) = 0$ for

$$S\varpi_- := A - \lambda/2 < 0 < S\varpi_+ := A + \lambda/2$$

with

$$\lambda = 2\sqrt{A^2 + RS}$$

$$\Downarrow$$

$\forall t \geq \upsilon > 0$

$$\left[|P_t - \varpi_+| \vee exp\left(2\int_0^t [A - P_s S]ds\right)\right] \leq c_\upsilon \ \exp\left(-\lambda t\right)$$

Stochastic Riccati flow $p_t \in \mathbb{R}_+$ with $\epsilon = 2/\sqrt{N}$:

$$dp_t \overset{law}{=} \mathrm{Ricc}(p_t)dt + \epsilon \sqrt{p_t(U + p_t V p_t)}\, dW_t$$

$$\text{with} \quad \epsilon^2 U < 2R \quad \Longrightarrow \quad \text{origin repellent}$$

Stochastic Riccati flow $p_t \in \mathbb{R}_+$ with $\epsilon = 2/\sqrt{N}$:

$$dp_t \overset{law}{=} \mathrm{Ricc}(p_t)dt + \epsilon \sqrt{p_t(U + p_t V p_t)}\; dW_t$$

$$\text{with} \quad \epsilon^2 U < 2R \quad \Longrightarrow \quad \text{origin repellent}$$

Reversible measures $\pi_\epsilon(dx)$ on $\mathbb{R}_+$:

- $U \wedge V > 0 \rightsquigarrow$ Heavy tails

$$\propto \quad \frac{x^{\frac{2}{\epsilon^2}\frac{R}{U}-1}}{[U + Vx^2]^{1+\frac{1}{\epsilon^2}\left(\frac{R}{U}+\frac{S}{V}\right)}}\ \exp\left[\frac{4}{\epsilon^2}\ \frac{A}{\sqrt{UV}}\ \tan^{-1}\left(x\sqrt{\frac{V}{U}}\right)\right] dx.$$

Stochastic Riccati flow $p_t \in \mathbb{R}_+$ with $\epsilon = 2/\sqrt{N}$:

$$dp_t \overset{law}{=} \mathrm{Ricc}(p_t)dt + \epsilon \sqrt{p_t(U + p_t V p_t)}\; dW_t$$

$$\text{with} \quad \epsilon^2 U < 2R \quad \Longrightarrow \quad \text{origin repellent}$$

Reversible measures $\pi_\epsilon(dx)$ on $\mathbb{R}_+$:

- $U \wedge V > 0 \rightsquigarrow$ Heavy tails

$$\propto \frac{x^{\frac{2}{\epsilon^2}\frac{R}{U}-1}}{[U + Vx^2]^{1+\frac{1}{\epsilon^2}\left(\frac{R}{U}+\frac{S}{V}\right)}} \exp\left[\frac{4}{\epsilon^2}\frac{A}{\sqrt{UV}} \tan^{-1}\left(x\sqrt{\frac{V}{U}}\right)\right] dx.$$

- $U > V = 0 \rightsquigarrow$ Gaussian-type tails

$$\propto x^{\frac{2}{\epsilon^2}\frac{R}{U}-1} \exp\left[-\frac{S}{U\epsilon^2}\left(x - 2\frac{A}{S}\right)^2\right] dx.$$

Stability Markov transition semigroup $\mathcal{P}_t^\epsilon$ (of p_t)

Th [+ `Bishop, Kamatani, Rémillard Arxiv-17`] $\forall A, R \wedge S > 0$

- $\exists \zeta, \epsilon_0 > 0$ and some Wasserstein distance $\mathbb{D}$ s.t. for any $0 \leq \epsilon \leq \epsilon_0$

$$\mathbb{D}(\mu_1 \mathcal{P}_t^\epsilon, \mu_2 \mathcal{P}_t^\epsilon) \leq \exp\left(-\lambda\ (1 - \epsilon^2 \zeta)\ t\right)\ \ \mathbb{D}(\mu_1, \mu_2)$$

Stability Markov transition semigroup $\mathcal{P}_t^{\epsilon}$ (of p_t)

Th [+ `Bishop, Kamatani, Rémillard Arxiv-17`] $\forall A, R \wedge S > 0$

- $\exists \zeta, \epsilon_0 > 0$ and some Wasserstein distance $\mathbb{D}$ s.t. for any $0 \leq \epsilon \leq \epsilon_0$

$$\mathbb{D}(\mu_1 \mathcal{P}_t^{\epsilon}, \mu_2 \mathcal{P}_t^{\epsilon}) \leq \exp\left(-\lambda\ (1 - \epsilon^2 \zeta)\ t\right)\ \ \mathbb{D}(\mu_1, \mu_2)$$

- $\forall n \geq 1\ \exists\ \zeta_n, \epsilon_n > 0$ for any $0 \leq \epsilon \leq \epsilon_n$

$$\mathbb{E}\left[\exp\left[n \int_0^t (A - p_s S) ds\right]\right]^{1/n} \leq c_Q\ \exp\left(-\lambda\ (1 - \epsilon^2 \zeta_n)\ t\right)$$

Stability Markov transition semigroup $\mathcal{P}_t^\epsilon$ (of p_t)

Th [+ `Bishop, Kamatani, Rémillard Arxiv-17`] $\forall A, R \wedge S > 0$

- $\exists \zeta, \epsilon_0 > 0$ and some Wasserstein distance $\mathbb{D}$ s.t. for any $0 \leq \epsilon \leq \epsilon_0$

$$\mathbb{D}(\mu_1 \mathcal{P}_t^\epsilon, \mu_2 \mathcal{P}_t^\epsilon) \leq \exp\left(-\lambda \left(1 - \epsilon^2 \zeta\right) t\right) \ \mathbb{D}(\mu_1, \mu_2)$$

- $\forall n \geq 1 \ \exists \ \zeta_n, \epsilon_n > 0$ for any $0 \leq \epsilon \leq \epsilon_n$

$$\mathbb{E}\left[\exp\left[n \int_0^t (A - p_s S) ds\right]\right]^{1/n} \leq c_Q \ \exp\left(-\lambda \left(1 - \epsilon^2 \zeta_n\right) t\right)$$

Some extensions

Case 2: Poincaré inequalities (and $\mathbb{L}_2(\pi_\epsilon)$-contractions), ...

Consequences

Uniform estimates for state estimates + particle Riccati diffusions,...

Kalman-Bucy filter

McKean-Vlasov interpretations

Stability of Kalman-Bucy diffusions
  - Stability of Riccati semigroup
  - Stability of stochastic flows

Uniform fluctuation estimates

Nonlinear models

Multivariate KB : Observability + Controllability

$$d(\widehat{X}_t - X_t)$$

$$= (\mathbf{A} - \mathbf{P_t S})\,(\widehat{X}_t - X_t)dt - \boldsymbol{R}^{1/2}\,\boldsymbol{d W_t} + \boldsymbol{P_t C' \Sigma}^{-1/2}\,\boldsymbol{d V_t}$$

Multivariate KB : Observability + Controllability

$$d(\widehat{X}_t - X_t)$$

$$= (\mathbf{A} - \mathbf{P_t S})\ (\widehat{X}_t - X_t)dt - \boldsymbol{R^{1/2}\ dW_t + P_t C' \Sigma^{-1/2}\ dV_t}$$

Steady state: $\exists! P_\infty > 0$ **s.t.** $\mathrm{Ricc}(P_\infty) = 0$ **and spectral abscissa**

$$\varsigma(\mathbf{A} - \mathbf{P_\infty S}) := \max\left\{\mathrm{Re}(\lambda)\ :\ \lambda \in \mathrm{Spec}(A - P_\infty S)\right\} < 0$$

Multivariate KB : Observability + Controllability

$$d(\widehat{X}_t - X_t)$$

$$= (\mathbf{A} - \mathbf{P_t S})\ (\widehat{X}_t - X_t)dt\ -\ \boldsymbol{R^{1/2}}\ \boldsymbol{d W_t} + \boldsymbol{P_t C' \Sigma^{-1/2}}\ \boldsymbol{d V_t}$$

Steady state: $\exists! P_\infty > 0$ **s.t.** $\mathrm{Ricc}(P_\infty) = 0$ **and spectral abscissa**

$$\varsigma(\mathbf{A} - \mathbf{P_\infty S}) := \max\{\mathrm{Re}(\lambda)\ :\ \lambda \in \mathrm{Spec}(A - P_\infty S)\} < 0$$

$$\Downarrow$$

STABLE EVEN WHEN *A* is unstable.

$\rightsquigarrow$ `SIAM Control & Opt.-17` $\oplus$ `Arxiv-18` (+ Bishop)
Review on the stability of Kalman-Bucy filters and Riccati matrix semigroups $\oplus$ *Floquet representation of exponential semigroups*

Floquet representations

$$P_t = \phi_t(P_0) \quad \text{flow of the Riccati equation} \quad \partial_t P_t = \mathrm{Ricc}(P_t)$$

$$\downarrow$$

Floquet representations

$$P_t = \phi_t(P_0) \quad \text{flow of the Riccati equation} \quad \partial_t P_t = \mathrm{Ricc}(P_t)$$

$$\downarrow$$

Exponential semigroups/Fundamental matrices:

$$E_{s,t}(P) = \exp \oint_s^t (A - \phi_u(P)S)\, du$$

Floquet representations

$$P_t = \phi_t(P_0) \quad \text{flow of the Riccati equation} \quad \partial_t P_t = \mathrm{Ricc}(P_t)$$

$$\downarrow$$

Exponential semigroups/Fundamental matrices:

$$E_{s,t}(P) = \exp \oint_s^t (A - \phi_u(P)S)\, du$$

in the sense that (with $E_{t,t}(P) = Id$)

$$\partial_t E_{s,t}(P) = (A - \phi_t(P)S) E_{s,t}(P) \quad \text{and} \quad \partial_s E_{s,t}(P) = -E_{s,t}(P)(A - \phi_s(P)S)$$

Floquet representations

$$P_t = \phi_t(P_0) \quad \text{flow of the Riccati equation} \quad \partial_t P_t = \mathrm{Ricc}(P_t)$$

$$\downarrow$$

Exponential semigroups/Fundamental matrices:

$$E_{s,t}(P) = \exp \oint_s^t (A - \phi_u(P)S)\, du$$

in the sense that (with $E_{t,t}(P) = Id$)

$$\partial_t E_{s,t}(P) = (A - \phi_t(P)S) E_{s,t}(P) \quad \text{and} \quad \partial_s E_{s,t}(P) = -E_{s,t}(P)(A - \phi_s(P)S)$$

Nb.:

$$P = P_\infty \Longrightarrow E_{s,t}(P_\infty) = e^{(t-s)(A - P_\infty S)} \quad \text{with} \quad A - P_\infty S \quad \text{stable}$$

Floquet representations 2/2

Theo.:`(+ Bishop - Arxiv-18)`

$$E_t(P) := \exp \oint_0^t \left(\mathbf{A} - \phi_{\mathbf{s}}(\mathbf{P})\mathbf{S}\right) ds = e^{t(A - P_\infty S)}\ \mathbb{C}_t(P)^{-1}$$

with

$$\sup_{t \geq 0} \|\mathbb{C}_t(P)^{-1}\| \leq c\ (1 + \|Q\|)$$

Cor.:

$$\|\phi_t(P_1) - \phi_t(P_2)\| \leq \|e^{t(A - P_\infty S)}\|\ (1 + \|P_1\|^2 + \|P_2\|^2)\ \|P_1 - P_2\|$$

⊕ **same type of estimates for the time varying linear process**

$$d(\widehat{X}_t - X_t)$$

$$= (\mathbf{A} - \mathbf{P_t S})\ (\widehat{X}_t - X_t)\ dt - \boldsymbol{R^{1/2}\ dW_t + P_t C' \Sigma^{-1/2}\ dV_t}$$

Kalman-Bucy filter

McKean-Vlasov interpretations

Stability of Kalman-Bucy diffusions

Uniform fluctuation estimates
    Some observations/numerical issues
    The Multivariate case

Nonlinear models

Back to the EnKF - Some observations/numerical issues

- $C = 0 \Rightarrow \xi_t^i$ i.i.d. copies of the signal $\Longrightarrow$ $p_t =$ Wishart process.

Back to the EnKF - Some observations/numerical issues

- $C = 0 \Rightarrow \xi_t^i$ i.i.d. copies of the signal $\Longrightarrow$ $p_t =$ Wishart process.
- $\mathrm{rank}(p_t) \leq N < r \Rightarrow (r - N)$ state dimensions not driven by Y_t.

Back to the EnKF - Some observations/numerical issues

- $C = 0 \Rightarrow \xi_t^i$ i.i.d. copies of the signal $\Longrightarrow$ $p_t =$ Wishart process.
- $\mathrm{rank}(p_t) \leq N < r \Rightarrow (r - N)$ state dimensions not driven by Y_t.
- **Case 1** : $r = 1 \Longrightarrow p_t$ heavy tail invariant distribution $\propto x^{-(N+3)}$

$$\Longrightarrow \quad \forall \epsilon > 0 \qquad \mathbb{E}(e^{\epsilon p_t}) = \infty \quad \text{and} \quad \forall m \geq N+2 \qquad \mathbb{E}(p_t^m) = \infty$$

$\rightsquigarrow$ **Moment explosions**

Back to the EnKF - Some observations/numerical issues

- $C = 0 \Rightarrow \xi_t^i$ i.i.d. copies of the signal $\Longrightarrow$ $p_t =$ Wishart process.
- $\mathrm{rank}(p_t) \leq N < r \Rightarrow (r - N)$ state dimensions not driven by Y_t.
- **Case 1** : $r = 1 \Longrightarrow p_t$ heavy tail invariant distribution $\propto x^{-(N+3)}$

$$\Longrightarrow \quad \forall \epsilon > 0 \qquad \mathbb{E}(e^{\epsilon p_t}) = \infty \quad \text{and} \quad \forall m \geq N+2 \qquad \mathbb{E}(p_t^m) = \infty$$

$\rightsquigarrow$ **Moment explosions**

- **Case 3** $=$ *Pure transport* $\rightsquigarrow$ simple estimates using Lipschitz and contraction inequalities w.r.t. initial conditions

Multivariate : EnKF

$$(m_t, X_t, \mathbf{p_t}) = (\text{sample mean}, \text{true signal}, \textbf{sample covariance})$$

$$\Downarrow$$

$$d(m_t - X_t) = (A - \mathbf{p_t} S)\,(m_t - X_t)\,dt - \boldsymbol{R}^{1/2} \boldsymbol{dW_t} + \boldsymbol{p_t}\ \boldsymbol{C'\Sigma^{-1/2}}\ \boldsymbol{dV_t} + \frac{\mathbf{d\overline{M}_t}}{\sqrt{\mathbf{N}}}$$

Multivariate : EnKF

$$(m_t, X_t, \mathbf{p_t}) = (\text{sample mean}, \text{true signal}, \textbf{sample covariance})$$

$$\Downarrow$$

$$d(m_t - X_t) = (A - \mathbf{p_t} S)\,(m_t - X_t)\,dt - \boldsymbol{R}^{1/2} \boldsymbol{dW_t} + \boldsymbol{p_t}\ \boldsymbol{C'\Sigma}^{-1/2}\ \boldsymbol{dV_t} + \frac{\mathbf{d\overline{M}_t}}{\sqrt{\mathbf{N}}}$$

Observations:

- **Time varying** $\oplus$ **stochastic type** Ornstein-Uhlenbeck diffusion

  DRIVEN BY A STOCH. MATRIX-RICCATI DIFFUSION $\mathbf{p_t}$

Multivariate : EnKF

$$(m_t, X_t, \mathbf{p_t}) = (\text{sample mean}, \text{true signal}, \textbf{sample covariance})$$

$$\Downarrow$$

$$d(m_t - X_t) = (A - \mathbf{p_t} S)\,(m_t - X_t)\,dt - \boldsymbol{R}^{1/2} \boldsymbol{dW_t} + \boldsymbol{p_t}\ \boldsymbol{C'\Sigma}^{-1/2}\ \boldsymbol{dV_t} + \frac{\mathbf{d\overline{M}_t}}{\sqrt{\mathbf{N}}}$$

Observations:

- **Time varying** $\oplus$ **stochastic type** Ornstein-Uhlenbeck diffusion

  DRIVEN BY A STOCH. MATRIX-RICCATI DIFFUSION $\mathbf{p_t}$

- The matrix $(A - pS)$ may be ill-conditioned in the sense that

  $\exists p\ :\ \lambda_{max}((A - pS)_{sym}) > 0$ even if A stable in dimension ≥ 2

Multivariate : EnKF

$$(m_t, X_t, \mathbf{p_t}) = (\text{sample mean}, \text{true signal}, \textbf{sample covariance})$$

$$\Downarrow$$

$$d(m_t - X_t) = (A - \mathbf{p_t} S)\,(m_t - X_t)\,dt - \boldsymbol{R^{1/2} dW_t} + \boldsymbol{p_t\ C'\Sigma^{-1/2}\ dV_t} + \frac{\mathbf{d\overline{M}_t}}{\sqrt{\mathbf{N}}}$$

Observations:

- **Time varying** $\oplus$ **stochastic type** Ornstein-Uhlenbeck diffusion

  DRIVEN BY A STOCH. MATRIX-RICCATI DIFFUSION $\mathbf{p_t}$

- The matrix $(A - pS)$ may be ill-conditioned in the sense that

  $$\exists p \ :\ \lambda_{max}((A - pS)_{sym}) > 0 \quad \text{even if } A \text{ stable in dimension} \geq 2$$

- Always under-biased

  $$\forall t > 0 \qquad 0 < p_t \quad \text{but} \quad 0 < \mathbb{E}(p_t) < P_t$$

Multivariate case

Hyp 1: $S > 0 \rightsquigarrow$ up a change of basis

$$(A, S) \rightsquigarrow (\overline{A}, \overline{S}) := (S^{1/2} A S^{-1/2}, I) \quad \rightsquigarrow \quad + \text{Hyp } 2 \quad \mu(\overline{A}) < 0$$

Multivariate case

Hyp 1: $S > 0 \rightsquigarrow$ **up a change of basis**

$$(A,S) \rightsquigarrow (\overline{A},\overline{S}) := (S^{1/2}AS^{-1/2}, I) \quad \rightsquigarrow \quad + \text{Hyp } 2 \quad \mu(\overline{A}) < 0$$

$$\Downarrow$$

Theo 1 `[+Tugaut AAP-17]` [Hyp 1 + Hyp 2] $\forall n \geq 1 \; \exists N_n \geq 1$:

$$N \geq N_n \Longrightarrow \sup_{t\geq 0}\left[\mathbb{E}\left(\|\boldsymbol{p_t} - \boldsymbol{P_t}\|^{\boldsymbol{n}}\right) \vee \mathbb{E}(\|m_t - \widehat{X}_t\|^n)\right] < c/\sqrt{N}$$

for the spectral of the Frobenius norm.

Multivariate case

Hyp 1: $S > 0 \rightsquigarrow$ **up a change of basis**

$$(A,S) \rightsquigarrow (\overline{A},\overline{S}) := (S^{1/2}AS^{-1/2}, I) \quad \rightsquigarrow \quad + \text{Hyp } 2 \quad \mu(\overline{A}) < 0$$

$$\Downarrow$$

Theo 1 `[+Tugaut AAP-17]` [Hyp 1 + Hyp 2] $\forall n \geq 1 \; \exists N_n \geq 1$:

$$N \geq N_n \Longrightarrow \sup_{t \geq 0} \left[\mathbb{E}\left(\|\boldsymbol{p_t} - \boldsymbol{P_t}\|^{\boldsymbol{n}}\right) \vee \mathbb{E}(\|m_t - \widehat{X}_t\|^n) \right] < c/\sqrt{N}$$

for the spectral of the Frobenius norm.

Theo 2 `[+Bishop et al. 17+18]` [only Observability + Controllability]

Uniform Riccati estimates "even" if we have the under-bias estimate

$$\forall t > 0 \qquad 0 < p_t \quad \text{but} \quad 0 < \mathbb{E}(p_t) < P_t$$

Multivariate case

Hyp 1: $S > 0 \rightsquigarrow$ up a change of basis

$$(A, S) \rightsquigarrow (\overline{A}, \overline{S}) := (S^{1/2} A S^{-1/2}, I) \quad \rightsquigarrow \quad + \text{Hyp 2} \quad \mu(\overline{A}) < 0$$

$$\Downarrow$$

Theo 1 `[+Tugaut AAP-17]` [Hyp 1 + Hyp 2] $\forall n \geq 1 \; \exists N_n \geq 1$:

$$N \geq N_n \Longrightarrow \sup_{t \geq 0} \left[\mathbb{E} \left(\| \boldsymbol{p_t} - \boldsymbol{P_t} \|^{\boldsymbol{n}} \right) \vee \mathbb{E}(\| m_t - \widehat{X}_t \|^n) \right] < c/\sqrt{N}$$

for the spectral of the Frobenius norm.

Theo 2 `[+Bishop et al. 17+18]` [only Observability + Controllability]

Uniform Riccati estimates "even" if we have the under-bias estimate

$$\forall t > 0 \qquad 0 < p_t \quad \text{but} \quad 0 < \mathbb{E}(p_t) < P_t$$

+ Same uniform rates+ Bias-Taylor type expansions + Robustness and Perturbations analysis (inflation, masking, shrinkage, projections,. . .),. . .

Nonlinear models

Extended Kalman-Bucy-filters

$$d\widehat{X}_t = A(\widehat{X}_t)\ dt + P_t C'\ \Sigma^{-1}\ \left[dY_t - C\widehat{X}_t\ dt\right]$$

with the "stochastic" Riccati equation:

$$\partial_t P_t = \partial A(\widehat{X}_t) P_t + P_t\ \partial A(\widehat{X}_t)' + R - P_t S P_t$$

Nonlinear models

Extended Kalman-Bucy-filters

$$d\widehat{X}_t = A(\widehat{X}_t)\ dt + P_t C'\ \Sigma^{-1}\ \left[dY_t - C\widehat{X}_t\ dt\right]$$

with the "stochastic" Riccati equation:

$$\partial_t P_t = \partial A(\widehat{X}_t) P_t + P_t\ \partial A(\widehat{X}_t)' + R - P_t S P_t$$

McKean-Vlasov interpretation

$$\begin{aligned} d\overline{X}_t \quad &= \quad \mathcal{A}\left(\overline{X}_t, \eta_t(e)\right)\ dt + R^{1/2}\ d\overline{W}_t \\ &\qquad + \mathcal{P}_{\eta_t} C' R_2^{-1}\ \left[dY_t - \left(C\overline{X}_t\ dt + \Sigma^{1/2}\ d\overline{V}_t\right)\right] \end{aligned}$$

with the drift function

$$\mathcal{A}(x, m) := A\left[m\right] + \partial A\left[m\right]\ (x - m).$$

Extended Ensemble Kalman-Bucy-filters

En-EKF = Mean field particle model

$$\begin{aligned} d\xi_t^i &= \mathcal{A}(\xi_t^i, m_t)\ dt + R^{1/2} d\overline{W}_t^i \\ &\qquad + p_t C' \Sigma^{-1} \left[dY_t - \left(C\xi_t^i\ dt + \Sigma^{1/2}\ d\overline{V}_t^i \right) \right] \end{aligned}$$

with the sample means m_t *and covariance matrices* p_t and the drift

$$\mathcal{A}(\xi_t^i, m_t) := A\left[m_t\right] + \underbrace{\partial A\left[m_t\right]\ (\xi_t^i - m_t)}_{\text{Repulsion/Attraction w.r.t. } m_t}$$

Some illustrations

Langevin type signal processes

$$R = \sigma^2 \; Id \quad \text{and} \quad (A, \partial A) = (-\partial \mathcal{V}, -\partial^2 \mathcal{V})$$

Non quadratic potential $(q \in \mathbb{R}^r, \mathcal{Q}_1, \mathcal{Q}_2 \geq 0)$

$$\mathcal{V}(x) = \frac{1}{2} \langle \mathcal{Q}_1 x, x \rangle + \langle q, x \rangle + \frac{1}{3} \langle \mathcal{Q}_2 x, x \rangle^{3/2}$$

Interacting diffusion gradient flows

$$\mathcal{V}(x) = \sum_{1 \leq i \leq r} \mathcal{U}_1(x_i) + \sum_{1 \leq i \neq j \leq r} \mathcal{U}_2(x_i, x_j)$$

for some convex confining potential $\mathcal{U}_i : \mathbb{R}^i \mapsto [0, \infty[$

Regularity conditions

Full observation $S = s\ Id$ and

$$\begin{aligned} -\lambda_{\partial A} &:= \sup_{x \in \mathbb{R}^r} \lambda_{max}(\partial A(x) + \partial A(x)') < 0 \\ \|\partial A(x) - \partial A(y)\| &\leq \kappa_{\partial A}\ \|x - y\| \end{aligned}$$

Examples: Langevin signal-diffusion

$$(\lambda_{\partial A}, \kappa_{\partial A}) = \beta\ \left(2^{-1}\lambda_{min}(\mathcal{Q}_1),\ 2\lambda_{max}^{3/2}(\mathcal{Q}_2)\right).$$

more generally $\partial^2\mathcal{V} \geq v\ Id \oplus$ *Lipschitz condition*

Stability theorem

$$(\overline{X}_t, \overline{Z}_t) := \text{McKean-Vlasov starting at} \quad (\overline{X}_0, \overline{Z}_0)$$

Stability theorem

$$(\overline{X}_t, \overline{Z}_t) := \text{McKean-Vlasov starting at} \quad (\overline{X}_0, \overline{Z}_0)$$

$$\Downarrow$$

Theo `[+Kurtzmann-Tugaut]`

When $\lambda_{\partial A}$ is sufficiently large we have

$$\mathbb{W}_2(\mathrm{Law}(\overline{X}_t), \mathrm{Law}(\overline{Z}_t)) \leq c \ \exp\left[-t \ \lambda\right] \quad \text{for some} \quad \lambda > 0.$$

$\exists$ *more explicit description in terms of* $(R, S, \kappa_{\partial A})$.

Propagation of chaos

$$\mathbb{P}_t^N := \mathrm{Law}(m_t, p_t) \qquad \mathbb{P}_t := \mathrm{Law}(\widehat{X}_t, P_t)$$

and

$$\mathbb{Q}_t^N := \mathrm{Law}(\xi_t^1) \qquad \mathbb{Q}_t := \mathrm{Law}(\overline{X}_t)$$

Propagation of chaos

$$\mathbb{P}_t^N := \mathrm{Law}(m_t, p_t) \qquad \mathbb{P}_t := \mathrm{Law}(\widehat{X}_t, P_t)$$

and

$$\mathbb{Q}_t^N := \mathrm{Law}(\xi_t^1) \qquad \mathbb{Q}_t := \mathrm{Law}(\overline{X}_t)$$

$$\Downarrow$$

Theo [+Kurtzmann-Tugaut]

When $\lambda_{\partial A}$ is sufficiently large, $\exists \beta \in]0, 1/2]$ s.t.

$$\sup_{t\geq 0} \mathbb{W}_2\left(\mathbb{P}_t^N, \mathbb{P}_t\right) \vee \sup_{t\geq 0} \mathbb{W}_2\left(\mathbb{Q}_t^N, \mathbb{Q}_t\right) \leq c\ N^{-\beta}$$

as soon as $\mathrm{tr}(P_0)$ is not too large and N large enough...