A Downscaling Data Assimilation Algorithm

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Question: How to make a weather forecast?

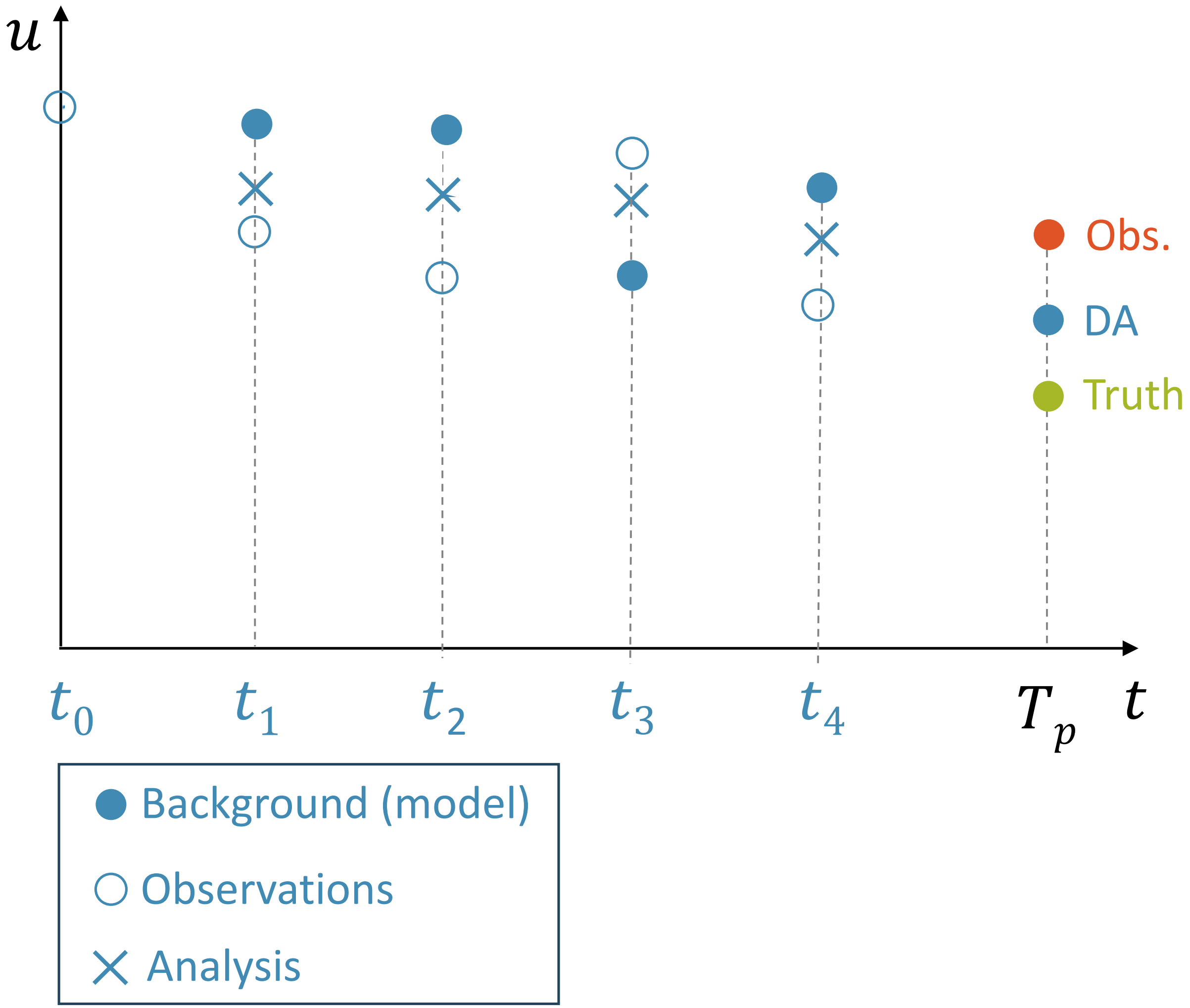
You will need...

- A theoretical model:

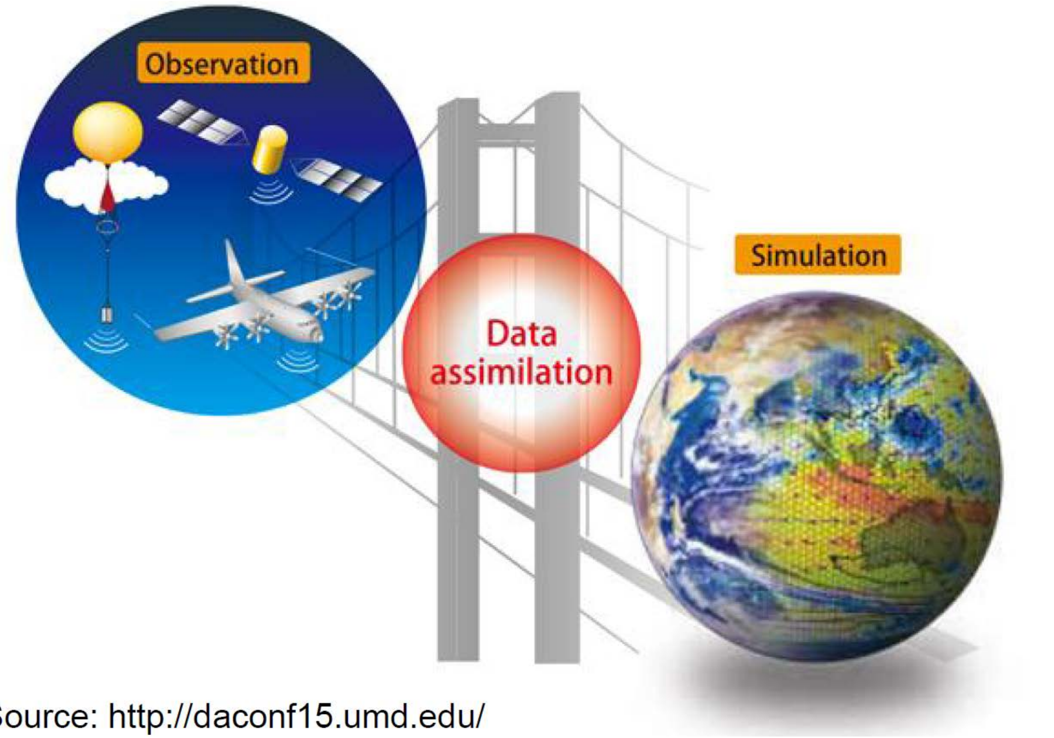
$$\frac{du}{dt} = F\big(t, u(t)\big)$$

u: unknown variable representing the state of the atmosphere (velocity field, temperature, moisture, ...).

- Observational measurements.



- ***Data Assimilation*** combines the theoretical model with information from observations in order to obtain a good approximation of the state of the physical system at a certain future time.



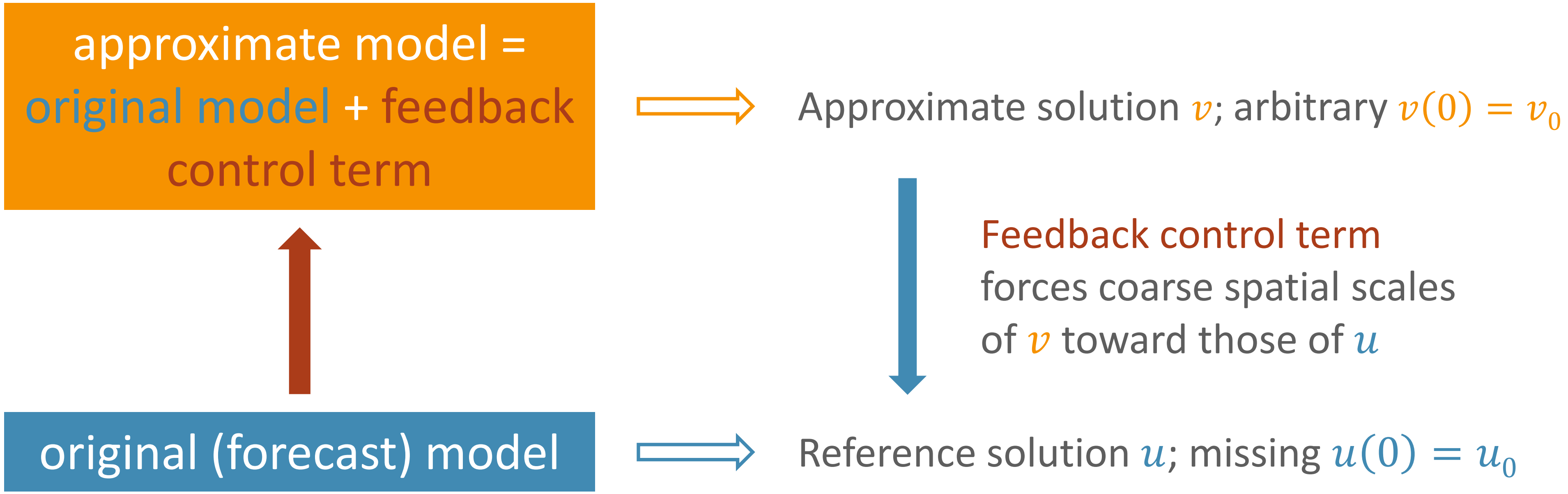
Source: http://daconf15.umd.edu/

- Numerous applications: meteorology, oceanography, oil industry, neuroscience, etc.

- Several approaches:
  - Nudging.
  - Kalman Filter (KF).
  - Ensemble Kalman Filter (EnKF).
  - Local Ensemble Transform Kalman Filter (LETKF).
  - 3DVAR.
  - 4DVAR.

Feedback-control (nudging) approach
(Azouani-Olson-Titi, ‘14)

- Combine model and measurements by adding a feedback-control term to the equations.



Background idea

- Long-time behavior of solutions to dissipative evolution equations is determined by only a ***finite*** number of degrees of freedom.
  - Fourier modes, 2D-NSE (Foias-Prodi, ‘67):

Let P_N be the projection operator onto the first N Fourier modes.
$\exists N \gg 1$ s.t. if $\mathbf{u}_1$, $\mathbf{u}_2$ are two solutions of 2D-NSE with

$$\|P_N \mathbf{u}_1 - P_N \mathbf{u}_2\|_{L^2} \to 0, \quad t \to \infty$$

then

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{L^2} \to 0, \quad t \to \infty.$$

  - Spatial nodes, 2D-NSE (Foias-Temam, ’84; Jones-Titi, ‘93).
  - Finite volume elements, 2D-NSE (Foias-Titi, ’91; Jones-Titi, ’92, ‘93).
  - Other dissipative evolution eqs. (Cockburn-Jones-Titi, ‘97).

Example

- Consider the forecast (theoretical) model given by the ***2D incompressible Navier-Stokes equations***:

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0 \qquad \text{(2D-NSE)}$$

$\mathbf{u}$: velocity field

ν : kinematic viscosity

p : pressure

$\mathbf{f}$: density of volume forces

- Assume:
  - No model error.
  - Continuous in time and error-free measurements.

Approximate model

controls small scales

controls large scales

$$\frac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\pi = \mathbf{f} - \beta[I_h(\mathbf{v}) - I_h(\mathbf{u})], \quad \nabla\cdot\mathbf{v} = 0.$$

ν, $\mathbf{f}$: same as for the 2D-NSE

π : modified pressure

h : resolution of spatial mesh

β : relaxation parameter

I_h : linear interpolant operator in space

- Denote $\mathbf{w} = \mathbf{v} - \mathbf{u}$.

$$\frac{\partial \mathbf{w}}{\partial t} - \nu \Delta \mathbf{w} + [(\mathbf{u}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{u} - (\mathbf{w}\cdot\nabla)\mathbf{w}] + \nabla(\pi - p) = -\beta I_h(\mathbf{w})$$
$$= -\beta[I_h(\mathbf{w}) - \mathbf{w}] - \beta \mathbf{w}$$

$$\Rightarrow \frac{\partial \mathbf{w}}{\partial t} - \nu \Delta \mathbf{w} + \beta \mathbf{w} + \nabla(\pi - p) = [(\mathbf{u}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{u} - (\mathbf{w}\cdot\nabla)\mathbf{w}] - \beta[I_h(\mathbf{w}) - \mathbf{w}]$$

- Assume

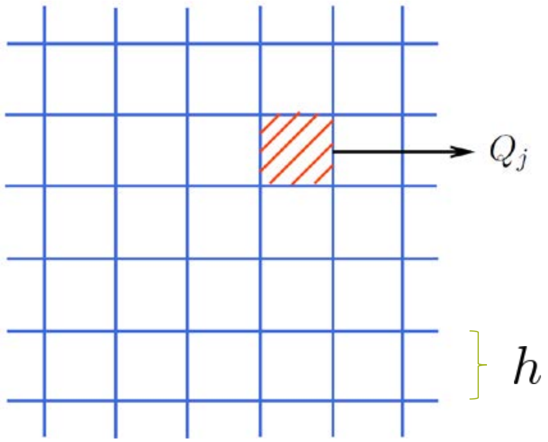
$$\|I_h(\varphi) - \varphi\|_{L^2} \le c_0 h \|\nabla \varphi\|_{L^2} \quad \forall \varphi \in (H^1)^2.$$

Example:

- Low modes projector: $I_h(\varphi) = P_N \varphi,\ N \in \mathbb{N}$.

- Finite volume elements: $\Omega = \bigcup_{j=1}^{N} Q_j$.

$$I_h(\varphi) = \sum_{j=1}^{N} \overline{\varphi_j}\chi_{Q_j}, \text{ where } \overline{\varphi_j} = \frac{1}{|Q_j|}\int_{Q_j} \varphi \mathrm{d}x.$$



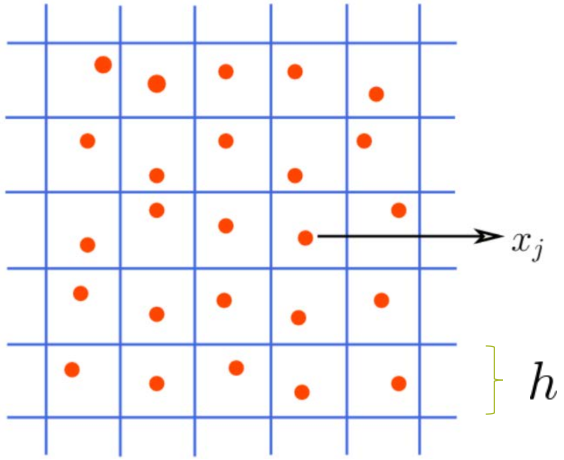
OR:

$$\|I_h(\varphi) - \varphi\|_{L^2} \leq c_0 h\|\varphi\|_{H^1} + c_1 h^2\|\varphi\|_{H^2} \quad \forall\varphi \in (H^2)^2.$$

Ex.:

- Nodal values: $x_j \in Q_j,\ j = 1, \ldots, N$.

$$I_h(\varphi) = \sum_{j=1}^{N} \varphi(x_j)\chi_{Q_j}.$$



Theorem (Azouani-Olson-Titi, ‘14)

If $\beta \gg \nu\lambda_1^2$ and $h \lesssim \nu^{1/2}/\beta^{1/2}$, then $\|\mathbf{v}(t) - \mathbf{u}(t)\| \leq O(\mathrm{e}^{-\beta t})$.

Some related works

- Other models: 3D NS-alpha (Albanez-Nussenzveig Lopes-Titi, ‘16), 3D Brinkman-Forchheimer-extended Darcy (Markowich-Titi-Trabelsi, ’16), 2D-SQG (Jolly-Martinez-Titi, ’17).
- Partial observations of the state variables:
  - 2D Bénard, only velocity (Farhat-Jolly-Titi, ’15).
  - 2D-NSE, one velocity component (Farhat-Lunasin-Titi, ’16) .
  - 3D planetary geostrophic model, only temperature (Farhat-Lunasin-Titi, ‘16).
  - 2D Bénard, only horizontal velocity component (Farhat-Lunasin-Titi, ’17).
  - 3D Bénard in porous media, only temperature (Farhat-Lunasin-Titi, ’17).
  - 3D Leray-alpha, only two components of velocity (Farhat-Lunasin-Titi, 17).

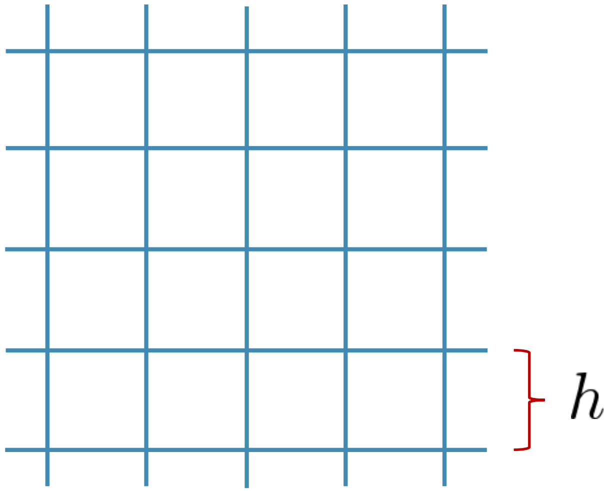
Some related works (cont'd)

- Higher order convergence, Gevrey class and L^∞ (Biswas-Martinez, '17).
- Measurements with stochastic errors (Blomker-Law-Stuart-Zygalakis, '13; Bessaih-Olson-Titi, '15).
- Time-averaged meas.: 2D-SQG (Jolly-Olson-Titi-Martinez), Lorenz (Blocher-Olson-Martinez).
- Numerical computations:
  - 2D-NSE (Gesho-Olson-Titi, '16).
  - 2D Bénard (Altaf-Titi-Gebrael-Knio-Zhao-McCabe-Hoteit, '16).
- Nonlinear continuous data assimilation – "super" exponential convergence (Larios-Pei, '17).
- Discrete in time meas. with syst. errors, 2D-NSE (Foias-Mondaini-Titi, '16).
- Numerical approximation by PPGM, 2D-NSE (Mondaini-Titi, '17).

Discrete in time Data Assimilation

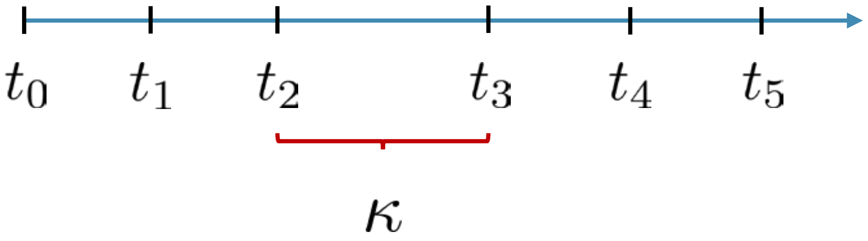
Measurements are...

- Discrete in space.



Spatial mesh with resolution of size h.

- Discrete in time.



$$|t_{n+1} - t_n| \leq \kappa, \quad n = 0, 1, 2, \ldots$$

- May contain errors.
  Denote by η_n the error at time t_n, $n = 0, 1, 2, \ldots$

Measurement at time t_n: $I_h(\mathbf{u}(t_n)) + \eta_n$

Approximate Model

$$\frac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\pi = \mathbf{f} - \beta\sum_{n=0}^{\infty}\{I_h(\mathbf{v}(t_n)) - [I_h(\mathbf{u}(t_n)) + \eta_n]\}\chi_{[t_n,t_{n+1})}$$

- Assume $I_h : (L^2)^2 \to (L^2)^2$ is a linear operator satisfying:

$$\|\varphi - I_h(\varphi)\|_{L^2} \le c_0 h\|\varphi\|_{H^1}, \quad \forall\varphi \in (H^1)^2.$$

$$\|I_h(\varphi)\|_{L^2} \le c_1\|\varphi\|_{L^2}, \quad \forall\varphi \in (L^2)^2.$$

- Examples: low Fourier modes projector, finite volume elements.

Theorem (Foias-Mondaini-Titi, '16)

Assume:

$$\|\eta_n\|_{H^1} \le E \quad \forall n.$$

If $\beta \gg \nu\lambda_1^2$, $\kappa \lesssim (\nu\lambda_1)^2/\beta^3$ and $h \lesssim \nu^{1/2}/\beta^{1/2}$, then

$$\limsup_{t\to\infty} \|\mathbf{v}(t) - \mathbf{u}(t)\|_{H^1} \le cE.$$

Moreover, if $E = 0$, then

$$\|\mathbf{v}(t) - \mathbf{u}(t)\|_{H^1} \le O(\mathrm{e}^{-\beta t}).$$

Numerical Approximation

- In practice, numerical models can only compute ***finite-dimensional*** approximations.
- ***Goal:*** Obtain an analytical estimate of the error between a numerical approximation of $\mathbf{v}$ and the (full) reference solution $\mathbf{u}$.
- For simplicity, assume: continuous in time and error-free measurements.
- Setting:
  - Phase space of 2D-NSE: $H = \{\mathbf{u} \in (L^2)^2 \,|\, \nabla \cdot \mathbf{u} = 0 \, + \, b.c.\}$.
  - Apply projector $P_\sigma : (L^2)^2 \to H$ to the feedback-control equation:

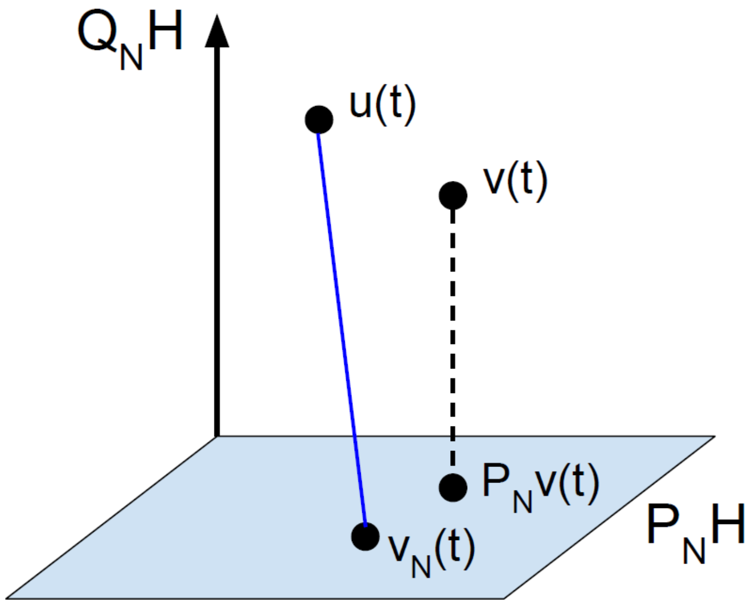
$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \nu A\mathbf{v} + B(\mathbf{v},\mathbf{v}) = \mathbf{f} - \beta P_\sigma I_h(\mathbf{v} - \mathbf{u}),$$

  - Eigenvectors of $A = P_\sigma(-\Delta)$: $\{\mathbf{w}_j\}_j$, with eigenvalues $\{\lambda_j\}_j$.
  - Finite-dimensional space: $\mathrm{span}\{\mathbf{w}_1, \ldots, \mathbf{w}_N\} = P_N H$.

Galerkin spectral method

Find $\mathbf{v}_N \in P_N H$ satisfying

$$\frac{\mathrm{d}\mathbf{v}_N}{\mathrm{d}t} + \nu A \mathbf{v}_N + P_N B(\mathbf{v}_N, \mathbf{v}_N) = P_N \mathbf{f} - \beta P_N P_\sigma I_h(\mathbf{v}_N - \mathbf{u}).$$



Notation: $Q_N = I - P_N.$

Theorem (Mondaini-Titi)

If $\beta \gg \nu\lambda_1^2$ and $h \lesssim \nu^{1/2}/\beta^{1/2}$, then $\exists\ \theta = \theta(\beta) \in [0,1)$ and $C = C(\nu, \lambda_1, |\mathbf{f}|_{L^2})$ s.t., for N sufficiently large,

$$\|\mathbf{v}_N(t) - \mathbf{u}(t)\|_{L^2} \leq c\theta^{(t-t_0)\nu\lambda_1 - 1}\|\mathbf{v}_N(t_0) - \mathbf{p}(t_0)\|_{L^2} + C\frac{L_N}{\lambda_{N+1}}.$$

Thus, $\exists\ T = T(\nu, \lambda_1, |\mathbf{f}|_{L^2}, N)$ s.t.

$$\|\mathbf{v}_N(t) - \mathbf{u}(t)\|_{L^2} \leq C\frac{L_N}{\lambda_{N+1}}, \quad \forall t \geq T,$$

where

$$L_N = \left[1 + log\left(\frac{\lambda_N}{\lambda_1}\right)\right]^{1/2}.$$

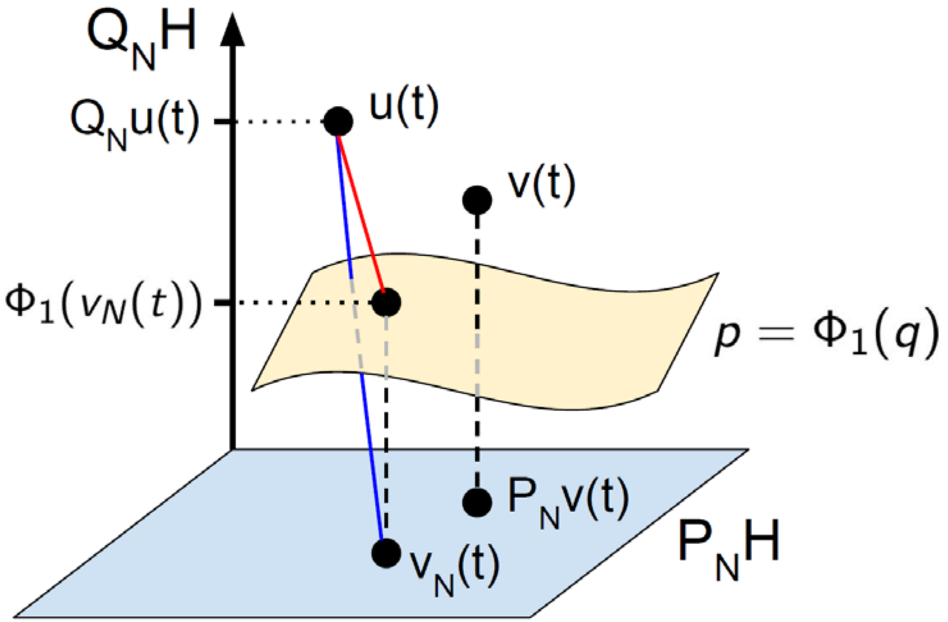
A Postprocessing of the Galerkin method

('García-Archilla'-Novo-Titi, '98)

- Idea: Add to the Galerkin approximation of $\mathbf{v}$ a suitable approximation of $\mathbf{q}$:

$$\mathbf{q} \approx \Phi_1(\mathbf{p}) = (\nu A)^{-1} Q_N[\mathbf{f} - B(\mathbf{p},\mathbf{p})]$$

(Approximate inertial manifold, Foias-Manley-Temam, '88)



Notation: $\mathbf{p} = P_N \mathbf{u}$, $\mathbf{q} = Q_N \mathbf{u}$

$(\mathbf{u} = \mathbf{p} + \mathbf{q})$

Postprocessing Galerkin Algorithm

For obtaining an approximation of $\mathbf{v}$, and thus $\mathbf{u}$, at a certain time $T > t_0$:

1. Integrate the Galerkin system over $[t_0, T]$ to obtain $\mathbf{v}_N(T)$.
2. Obtain $\mathbf{q}_N$ satisfying $\nu A\mathbf{q}_N = Q_N[\mathbf{f} - B(\mathbf{v}_N(T), \mathbf{v}_N(T)]$.
3. Compute $\mathbf{v}_N(T) + \mathbf{q}_N$.

- Information on the high modes (fine spatial scales) is only used at the final time T! This is one of the reasons for the efficiency of the Postprocessing Galerkin method (compared to, e.g., the Nonlinear Galerkin method).

Particular case: $I_h = P_K,\ K \in \mathbb{N}$

Theorem (Mondaini-Titi)

If $\beta \gg \nu\lambda_1^2$ and $\lambda_K \gtrsim \beta/\nu$, then $\exists\ \theta = \theta(\beta) \in [0,1)$ and $C = C(\nu, \lambda_1, |\mathbf{f}|_{L^2})$ s.t., for N sufficiently large,

$$\|(\mathbf{v}_N(t) + \Phi_1(\mathbf{v}_N(t)) - \mathbf{u}(t)\|_{L^2} \leq c\theta^{(t-t_0)\nu\lambda_1 - 1}\|\mathbf{v}_N(t_0) - \mathbf{p}(t_0)\|_{L^2} + C\frac{L_N^4}{\lambda_{N+1}^{3/2}}.$$

Thus, $\exists\ T = T(\nu, \lambda_1, |\mathbf{f}|_{L^2}, N)$ s.t.

$$\|(\mathbf{v}_N(t) + \Phi_1(\mathbf{v}_N(t))) - \mathbf{u}(t)\|_{L^2} \leq C\frac{L_N^4}{\lambda_{N+1}^{3/2}}, \quad \forall t \geq T.$$

General case

- Assume $I_h : (L^2)^2 \to (L^2)^2$ is a linear operator satisfying:
  - $\exists\, c_0 > 0$ s.t.
    $$\|\varphi - I_h(\varphi)\|_{L^2} \leq c_0 h \|\varphi\|_{H^1}, \quad \forall \varphi \in H^1(\Omega)^2.$$
  - $\exists\, c_{-1} > 0$ s.t.
    $$\|\varphi - I_h(\varphi)\|_{H^{-1}} \leq c_{-1} h \|\varphi\|_{L^2}, \quad \forall \varphi \in L^2(\Omega)^2.$$
  - $\exists\, \tilde{c}_0 > 0$ s.t.
    $$\|I_h(\mathbf{q})\|_{L^2} \leq \widetilde{c_0} \frac{|\Omega|^{3/4}}{h^2 \lambda_{N+1}^{1/4}} \|\mathbf{q}\|_{L^2}, \quad \forall \mathbf{q} \in Q_N H.$$
- Examples: low Fourier modes projector; finite volume elements.

Theorem (Mondaini-Titi [SIAM J. NUM. Anal. 2018])

If $\beta \gg \nu\lambda_1^2$ and $h \lesssim \nu^{1/2}/\beta^{1/2}$, then $\exists\ \theta = \theta(\beta) \in [0,1)$ and $C = C(\nu, \lambda_1, |\mathbf{f}|_{L^2})$ s.t., for N sufficiently large,

$$\|(\mathbf{v}_N(t) + \Phi_1(\mathbf{v}_N(t))) - \mathbf{u}(t)\|_{L^2} \leq c\theta^{(t-t_0)\nu\lambda_1 - 1}\|\mathbf{v}_N(t_0) - \mathbf{p}(t_0)\|_{L^2} + C\frac{L_N}{\lambda_{N+1}^{5/4}}.$$

Thus, $\exists\ T = T(\nu, \lambda_1, |\mathbf{f}|_{L^2}, N)$ s.t.

$$\|(\mathbf{v}_N(t) + \Phi_1(\mathbf{v}_N(t))) - \mathbf{u}(t)\|_{L^2} \leq C\frac{L_N}{\lambda_{N+1}^{5/4}}, \quad \forall t \geq T.$$

Comparison

- Error using the Galerkin method (both types of I_h):

$$\|\mathbf{v}_N - \mathbf{u}\|_{L^2} \leq O(L_N \lambda_{N+1}^{-1}).$$

- Error using the Postprocessing Galerkin method:
  - Case $I_h = P_K$:

$$\|(\mathbf{v}_N + \Phi_1(\mathbf{v}_N)) - \mathbf{u}\|_{L^2} = O(L_N^4 \lambda_{N+1}^{-3/2}).$$

  - General class of I_h :

$$\|(\mathbf{v}_N + \Phi_1(\mathbf{v}_N)) - \mathbf{u}\|_{L^2} = O(L_N \lambda_{N+1}^{-5/4}).$$

- Error estimates are uniform in time – feedback-control term stabilizes the large scales of the difference v – u, resulting in a globally asymptotically stable system.

Recent extensions

- [Ibdah-Mondaini-Titi 2018] Similar results for fully discrete systems.

- [Garcia-Archilla, Novo & Titi 2018] Similar results for the finite elements version.

Thank you!