Topics in Mathematical Imaging
Lecture 4

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Lecture plan

- Lecture 1: Variational models & PDEs for imaging by examples
- Lecture 2: Derivation of these models & analysis
- Lecture 3: Numerical solution
- Lecture 4: Some machine learning connections
The variational approach

General task: **restore** $u$ from an **observed datum** $g$ where

$$g = \underbrace{Tu}_{\text{forward model}} + \underbrace{n}_{\text{noise}}.$$ 

**Variational approach:** Compute $u$ as a minimizer of

$$\mathcal{J}(u) = \alpha \underbrace{R(u)}_{\text{regularization}} + \underbrace{D(Tu, g)}_{\text{data fidelity}} \to \min_{u \in B}.$$ 

where

- $R(u)$ is a prior/regularizer that models a-priori information on $u$ weighted by positive $\alpha$, e.g., $R(u) = \|\nabla u\|_{L^1}$
- $D(\cdot, \cdot)$ is a distance function, e.g. $D(Tu, g) = \|Tu - g\|_2^2$ and $B$ suitable Banach space, e.g., $B = BV(\Omega)$.

Engl, Hanke, Neubauer ’96; Natterer, Wübbeling ’01; Kaltenbacher, Neubauer, Scherzer ’08; Schuster, Kaltenbacher, Hofmann, Kazimierski ’12
Which model to choose?

Mathematics can make you fly! J. Grah, K. Papafitsoros, CBS, EPSRC Science Photo Award ’14, Burger, He, CBS ’09; CBS, Bertozzi ’11; CBS, CUP ’15; Chan, Shen ’01; Bertalmio et al. ’00; Masnou, Morel ’98.
Diffusion versus transport inpainting

Input image

References: Bertalmio, Sapiro, Caselles, Ballester 2000; Telea 2004; Bornemann, Maerz 2007; Burger, He, CBS, SIAM Imaging Science '09; CBS, CUP '15.
Diffusion versus transport inpainting

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Transport

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How to inpaint?

Image inpainting: create desired inpaintings.

‘Ecce mono’
How to inpaint?

Image inpainting: create desired inpaintings.

‘Ecce homo’

‘Ecce mono’
How to inpaint?

Image inpainting: create desired inpaintings.

Image courtesy of R. Hocking.

References: Arias, Facciolo, Caselles, Sapiro ’09–

‘Ecce mono’
Learning variational models - one idea

Assumptions

Training set of pairs \((f_k, u_k), \ k = 1, \ldots, N\) with
- \(f_k\) imperfect data
- \(u_k\) represent the ground truth

Determine optimal regulariser \(R\), data model \(\phi\), and \(\alpha\) in admissible set \(\mathcal{A}\)

\[
\min_{(R,\phi,\alpha,T)\in \mathcal{A}} \sum_k \text{loss}(\bar{u}_k, u_k)
\]

subject to

\[
\bar{u}_k = \arg\min_u \left\{ \alpha \ R(u) + \int_{\Omega} \phi(Tu, f_k) \, dx \right\}
\]
Learning by optimisation in imaging

Some contributions

- **Odone '05–, Tappen et al. '07, '09; Domke '11–**: Markov Random Field models; stochastic descent method.
- **Lui, Lin, Zhang and Su '09**: optimal control approach, no analytical justification; promising numerical results.
- **Horesh, Tenorio, Haber et al. '03–**: optimal design; $\ell_1$ minimisation.
- **Kunisch and Pock '13, Pock 13'**: results for finite dimensional case; optimal image filters; optimal SVM; optimal reaction-diffusion . . .
- **De Los Reyes, CBS '13**: results on bilevel learning in function space and development of numerical optimisation.
- **Fornasier, Naumova, Pereverzyev 14’**: parameter estimation in multipenalty regularisation.
- **Hintermüller et al. ’14**: bilevel optimisation for blind deconvolution, and for adaptive TV denoising.
- **Nikolova, Steidl, Weiss ’15**
- **Fonseca, Liu et al. ’16**: bilevel model for higher-order TV type regularisation and Mumford-Shah; analysis in function space . . .
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Analysis in function space & resolution independent optimisation.
Learning a parametrised model

Look for \( \lambda = (\lambda_1, \ldots, \lambda_M) \) and \( \alpha = (\alpha_1, \ldots, \alpha_N) \) solving

\[
\min_{(\lambda, \alpha) \in [0, \infty]^{M+N}} \int F(u_{\lambda, \alpha})
\]

subject to

\[
ul, \alpha \in \arg\min_{u \in X} \sum_{i=1}^{M} \int_{\Omega} \lambda_i(x) \phi_i([Tu](x)) \, dx
\]

\[
\sum_{j=1}^{N} \int_{\Omega} \alpha_j(x) \, d|A_ju|(x).
\]

Here \( T : X \rightarrow Y \subset L^1(\Omega; \mathbb{R}^d) \) with \( X, Y \) Banach spaces, \( A_j : X \rightarrow \mathcal{M}(\Omega; \mathbb{R}^{m_j}) \), \( j = 1, \ldots, N \) are appropriate linear operators, \( |A_ju| \) total variation measure, \( F \) is cost function.
Cross-validated computations on the Berkeley database split into two halves (100 images each):
Total variation regularisation with $L^2$ cost and fidelity. Noise variance $\sigma = 10$.

<table>
<thead>
<tr>
<th>Validation</th>
<th>Learning</th>
<th>$\alpha$</th>
<th>Average PSNR</th>
<th>Average SSIM</th>
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<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.0190</td>
<td>31.3679</td>
<td>0.8885</td>
</tr>
<tr>
<td>1</td>
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<tr>
<td>2</td>
<td>2</td>
<td>0.0190</td>
<td>31.2612</td>
<td>0.8850</td>
</tr>
</tbody>
</table>
Parameter optimality?

Quality measure

- Original cost functional (left figure) \( \| u - u_k \|_{L^2}^2 \)
- Signal to noise ratio (right figure)

\[
SNR = 20 \times \log_{10} \left( \frac{\| u_k \|_{L^2}}{\| u - u_k \|_{L^2}} \right),
\]
Parameter optimality?

(A) $u_c$ in blue and $u_\eta$ in red

(B) $\mathcal{I}(\alpha)$ is not quasi-convex

Courtesy of Pan Liu and Irene Fonseca using Strong, Chan, et al. '96.
And that is not all . . .

A few more examples of bringing together model-based imaging and learning . . .
Let us consider the following nonconvex model [Roth, Black '09], [Samuel, Tappen '09], called the “Fields of Experts” model:

$$\mathcal{R}(u) = \sum_{k=1}^{q} \sum_{i,j=1}^{m,n} \rho_k((K_k u)_{i,j})$$

- \(\{K_k\}\) are arbitrary filter kernels, and \(\{\rho_k\}\) are potential functions.
- Has much more parameters compared to the \(\ell_1\) model (several thousands).
- Allows only to compute a stationary point (local minimum).
- Suitable potential functions \(\rho_1\) are derived from statistics of natural images [Huang and Mumford '99]:

$$\rho_k(t) = \alpha_k \log(1 + \beta_k t^2)$$
The learned filters and functions

- In [Chen, Ranftl, P. ’14] we learned 80 filters of size $9 \times 9$ plus function parameters → 6480 parameters on a database of ~ 200 images
- ... two weeks later ...
Evaluation

- Comparison with five state-of-the-art approaches: K-SVD [Elad and Aharon '06], FoE [Q. Gao and Roth '12], BM3D [Dabov et al. '07], GMM [D. Zoran et al. '12], LSSC [Mairal et al. '09]

- We report the average PSNR on 68 images of the Berkeley image data base [Chen, P. 14]

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>KSVD</th>
<th>FoE</th>
<th>BM3D</th>
<th>GMM</th>
<th>LSSC</th>
<th>BL7x7</th>
<th>BL9x9</th>
</tr>
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<tbody>
<tr>
<td>15</td>
<td>30.87</td>
<td>30.99</td>
<td>31.08</td>
<td>31.19</td>
<td><strong>31.27</strong></td>
<td>31.18</td>
<td>31.22</td>
</tr>
<tr>
<td>50</td>
<td>25.17</td>
<td>25.35</td>
<td>25.62</td>
<td>25.67</td>
<td>25.72</td>
<td>25.70</td>
<td><strong>25.76</strong></td>
</tr>
</tbody>
</table>

- Performs equally or better as the state-of-the-art
Variational networks

- Inspired by the conditional shrinkage fields (CSF) [Schmidt, Roth '14], we allow to change the parameters during the iterations:

\[
\begin{aligned}
    u^0 &= f \\
    u^{t+1} &= u^t - \lambda^t \left( \sum_{k=1}^{q} (K^t_k)^\top (\rho^t_k)'(K^t_k u^t) + (u^t - f) \right), \quad t = 0 \ldots T - 1
\end{aligned}
\]

- In each step we perform one gradient descent on a learned variational energy

- Can be interpreted as one cycle of a block incremental gradient descent

- Can also be interpreted as learned non-linear diffusion, trying to “invert” the convolution \( \int p(f|u)p(u)du \)

- And it can be interpreted as a convolutional neural network with \( T \) layers
Quantitative evaluation

- We evaluated our learned models on a standard database of 68 images

<table>
<thead>
<tr>
<th>Method</th>
<th>$\sigma = 15$</th>
<th>St.</th>
<th>$\sigma = 25$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma = 15$</td>
<td></td>
<td>$\sigma = 25$</td>
</tr>
<tr>
<td>BM3D</td>
<td>31.08</td>
<td>15</td>
<td>31.14</td>
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<tr>
<td>LSSC</td>
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<td>25</td>
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<td>31.34</td>
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<tr>
<td>opt-MRF</td>
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<td>8</td>
<td>31.43</td>
</tr>
<tr>
<td>RTF$_5$</td>
<td>31.18</td>
<td>-</td>
<td>31.43</td>
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<tr>
<td>WNNM</td>
<td>31.37</td>
<td>2</td>
<td>28.58</td>
</tr>
<tr>
<td>CSF$_{5\times5}$</td>
<td>31.14</td>
<td>5</td>
<td>28.78</td>
</tr>
<tr>
<td>CSF$_{7\times7}$</td>
<td>31.24</td>
<td>8</td>
<td>28.83</td>
</tr>
</tbody>
</table>
Learning to reconstruct

- Variational regularization: Iterative schemes
- Learned operators
- Data in $\rightarrow$ reconstruction out

**Algorithm 1** Learned Gradient

1. for $i = 1, \ldots$ do
2. \[ \Delta f_i \leftarrow \Lambda_\Theta (f_i, \nabla [L(T(\cdot), g)](f_{i-1})) \]
3. \[ f_i \leftarrow f_{i-1} + \Delta f_i \]

Ground truth                  FBP (36 dB)

TV (38 dB)                    Learned (44 dB)

Deep image processing

This is unfeasible for many ill-posed inverse imaging problems

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Thank you very much for your attention!

More information see:
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