Lecture 3

Estimation of parameters

\[ P^{\lambda, \beta} = \frac{1}{Z^{\lambda, \beta}} e^{-\beta H} \pi^\Lambda \]

\( \gamma^* \) a realization of \( P^{\lambda, \beta} \) in \( \Lambda \)

\( \gamma^* \beta^* \) are unknown.

**aim:** \( \hat{\lambda}, \hat{\beta} \) estimates of \( \gamma^*, \beta^* \)

**condition:** \( H^\theta, \theta \in \Theta \)

- \( H^\theta = \theta_1 H_1 + \ldots + \theta_k H_k \) (linear case)
- non linear case

**asymptotic:** \( \Lambda \to \mathbb{R}^d \)

I) MLE

reformulated:

\[ P^{\lambda, \beta} = \frac{1}{Z^{\lambda, \beta}} e^{-\beta H} M_{\Lambda} \pi^\Lambda \]

**Definition:** MLE is defined as

\[ \hat{\lambda}, \hat{\beta} = \arg\max_{\lambda, \beta} \frac{1}{Z^{\lambda, \beta}} e^{-\beta H(\gamma^* \beta^*)} M_{\Lambda}(\beta^*) \]

**Theorem with assumption assumptions:** \( \hat{\lambda}, \hat{\beta} \) as \( \Lambda \to \mathbb{R}^d (\gamma^*, \beta^*) \) - Leavens
Remark: Identifiability $\iff$ Variational Principle.

- No result for normality
  - Untractable $\in \Lambda$

+ Seems to the better
  - Efficient $\hat{\alpha}, \hat{\beta}$ are small (Hose 92)

II) Terasco–Fiksel Estimator (84)

GNZ equations.

$p: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^+$

\[
C_{\Lambda}^{\beta, \beta}(\hat{\alpha}, \hat{\beta}) = \sum_{x \in \Lambda} \int p(x, \delta \lambda x) - 3 \int_{x \in \Lambda} \delta E^{-\delta}(x, \delta) \delta x
\]

\[
\Rightarrow E_{p_{\beta, \beta}^*(\Lambda)} \left( C_{\Lambda}^{\beta, \beta}(\hat{\alpha}, \hat{\beta}) \right) = 0
\]

TFE = mean square procedure

**Definition**

$K \geq 2$, $(\hat{\alpha}_i)_{1 \leq i \leq K}$ functions

\[
(\hat{\alpha}, \hat{\beta}) = \min_{(\alpha, \beta)} \sum_{k=1}^{K} \left( C_{\Lambda}^{\beta, \beta}(\hat{\alpha}_k, \delta) \right)^2
\]

**Theorem:** Under Ass

$\hat{\alpha}, \hat{\beta}$ is consistent and Ass Normal

(Coupselly, Der, Drouilhet, Lovacier)
identifiability Ass:
- Variance is larger than HLE
- optimization problem

Theoretical results:
- flexible
- no partition function

Identifiability Ass:
\[
E \left( \frac{p_h(0, r)}{r} \left[ g e^{-p_h(0, r)} - g^* e^{-p^* h(0, r)} \right] \right) = 0
\]

\( \forall h \)

\( \beta = \beta^* \) and \( \gamma = \gamma^* \)

\( h = 3 \)

\( \mathcal{H}(\gamma) \)

\( f(\gamma) = 1 \), \( f(\gamma) = h(x, y) \)

We recover the PMLE
(GNZ) \quad = \beta E_{P, \varphi, \rho} \left( \sum_{x \in \mathbb{R}^n} \nabla_x f(x, \varphi x) \right)
Definition: The variational Estimator
\[ \hat{\beta} = \frac{\sum_{x \in \mathbb{X}_n} \text{div}_x f(x, \beta' | x)}{\sum_{x \in \mathbb{X}_n} f(x, \beta' | x) \text{div}_x h(x, \beta' | x)} \]

Theorem: consistency and normality
(Puldeley - Der.)
\[ f(x, \delta) = \text{div}_x f(x, \delta) \]

- \( \hat{\beta} \) is not estimated
  - larger variance than the MLE and PHLE
- Theoretical results
  - flexible procedure
  - simple to compute.